Advanced Macroeconomics I

Lecture 2 (4)

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Spring 2010

Dynamic Programming

- The sequencial approach is to look for a sequence of real numbers $\{k_{t+1}\}_{t=0}^{\infty}$ that generates an optimal consumption plan
- In principle, this involved searching for a solution to an infinite sequence of equations - a difference equation (the Euler equation)
- The search for a sequence is sometimes impractical, and not always intuitive
- An alternative approach is often available, however, one which is useful conceptually as well as for computation. It is called dynamic programming

Dynamic Programming

- Key to dynamic programming is to think of dynamic decisions as being made not once and for all but recursively: time period by time period
  - The savings between t and $t + 1$ are thus decided on at t, and not at 0

Stationary

in terms of decisions

- We will call a problem **stationary** whenever the *structure* of the choice problem that a decision maker faces is *identical* at every point in time
- A consumer placed at the beginning of time choosing his infinite future consumption stream given an initial capital stock k_0
  - $\{k_{t+1}\}_{t=0}^{\infty}$ indicating the level of capital that the agent will choose to hold in each period
- But once he has chosen a capital path, suppose that we let the consumer abide it for T periods. At t = T he will find then himself with the k_T decided on initially. If at that moment we told the consumer to forget about his initial plan and asked him to decide on his consumption stream again, from then onwards, using as new initial level of capital $k_0 = k_T$, what sequence of capital would he choose?
- If the problem is stationary then for any two periods $t \neq s$, $k_t = k_s$, $k_{t+j} = k_{s+j}$, for all j > 0. That is, he would not change his mind if he could decide all over again

Decision Rule

- This means that, if a problem is stationary, we can think of a function that, for every period t; assigns to each possible initial level of capital k_t an optimal level for next period's capital k_{t+1} (and therefore an optimal level of current period consumption): $k_{t+1} = g(k_t)$
- Stationarity means that the function $g(k_t)$ has no other argument than current capital. In particular, the function does not vary with time. We will refer to $g(k_t)$ as the decision rule

Stationary

in terms of properties of the solution to a dynamic problem

- What types of dynamic problems are stationary?
  Intuitively, a dynamic problem is stationary if one can capture all relevant information for the decision maker in a way that does not involve time
  - In our neoclassical growth framework, with a finite horizon, time is important, and the problem is not stationary: it matters how many periods are left - the decision problem changes character as time passes
  - With an infinite time horizon, however, the remaining horizon is the same at each point in time. The only changing feature of the consumer's problem in the infinite-horizon neoclassical growth economy is his initial capital stock; hence, his decisions will not depend on anything but this capital stock
  - Whatever is the relevant information for a consumer solving a dynamic problem, we will refer to it as his state variable. So the state variable for the planner in the one-sector neoclassical growth context is the current capital stock

Value function

Let us denote by $V(k_t)$ the value of the optimal program from period t for an initial condition k_t:

$$V(k_t) \equiv \max_{\{k_{s+1}\}_{s=t+1}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} F(k_s, k_{s+1}) \quad \text{s.t. } k_{s+1} \in \Gamma(k_s) \ \forall s \geq t$$

where $\Gamma(k_s)$ represents the feasible choice set for k_{s+1} given k_s. That is, V is an indirect utility function, with k_t representing the parameter governing the choices and resulting utility. Then using the maximization-by-steps idea, we can write

$$V(k_t) \equiv \max_{k_{t+1} \in \Gamma(k_t)} \{F(k_t, k_{t+1}) + \beta V(k_{t+1})\}$$

Bellman equation

A functional equation – the unknown is a function

$$V(k) \equiv \max_{k' \in \Gamma(k)} \{F(k, k') + \beta V(k')\}$$

The decision rule for $k' : k' = g(k)$

$$g(k) = \arg \max_{k' \in \Gamma(k)} \{F(k, k') + \beta V(k')\}$$

The equivalence

between the sequential formulation of the dynamic optimization and its recursive, Bellman formulation

- If a function represents the value of solving the sequential problem (for any initial condition), then this function solves the dynamic programming equation (DPE)
- If a function solves the DPE, then it gives the value of the optimal program in the sequential formulation
- If a sequence solves the sequential program, it can be expressed as a decision rule that solves the maximization problem associated with the DPE
- If we have a decision rule for a DPE, it generates sequences that solve the sequential problem