Advanced Macroeconomics I

Lecture 2 (5)

Zhe Li

SUFE

Spring 2010

Existence of a value function

The proofs of all these results can be found in Stokey and Lucas with Prescott (1989)

Assumptions:

1. $F(k, k')$ is continuously diffrentiable in its two arguments, that it is strictly increasing in its first argument (and decreasing in the second), strictly concave, and bounded
2. Γ is a nonempty, compact-valued, monotone, and continuous correspondence with a convex graph
3. $\beta \in (0, 1)$

- There exists a function V that solves the Bellman equation. This solution is unique

Find the value function

Contraction Mapping

- It is possible to find V by the following iterative process:
  - Pick any initial V_0 function, for example $V_0(k) = 0$
  - Find V_{n+1}, for any value of k, by evaluating the right-hand side of (*) using V_n

$$V(k) \equiv \max_{k' \in \Gamma(k)} \{F(k, k') + \beta V(k')\} \quad (*)$$

- The outcome of this process is a sequence of functions $\{V_j\}_{j=0}^{\infty}$ which converges to V

Properties of a value function

- V is strictly concave
- V is strictly increasing
- V is diffrentiable

Optimal decision rule

- Optimal behavior can be characterized by a function g, with $k' = g(k)$, that is increasing so long as F_2 is increasing in k
- First order condition:

$$V(k) \equiv \max_{k' \in \Gamma(k)} \left\{ F(k, k') + \beta V(k') \right\} \tag{*}$$

$$-F_2(k, k') = \beta V'(k')$$