Advanced Macroeconomics I

Lecture 3 (1)

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Steady States

one-sector optimal growth model

$$\max_{c_t,\ k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$c_t + k_{t+1} = f(k_t)$$

$$f(k_t) = F(k_t, N) + (1-\delta)k_t$$

Euler equation:

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1})$$

Steady States

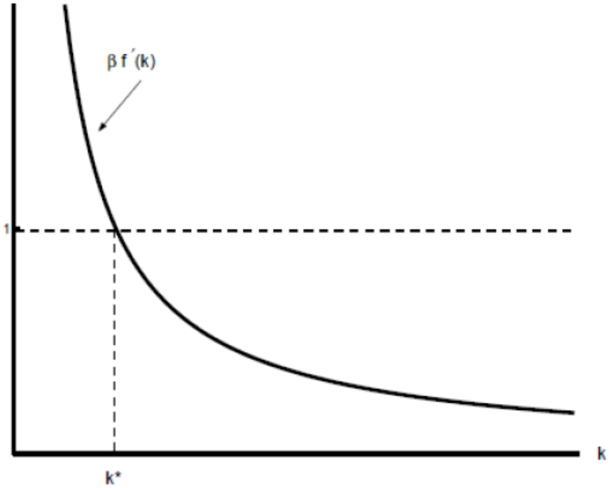
A steady state is a "constant solution":

$$
\begin{aligned}
k_t &= k^*, \forall t \\
c_t &= c^*, \forall t
\end{aligned}
$$

$$
\begin{aligned}
u'(c^*) &= \beta u'(c^*) f'(k^*) \\
\beta f'(k^*) &= 1
\end{aligned}
$$

It requires that the gross marginal productivity of capital equal the gross discount rate $\frac{1}{\beta}$

Existence



Optimal

Suppose $k_0 = k^*$, whether $k_t = k^*$ will solve the maximization problem?

- By construction, first order conditions are satisfied
- Transversality condition?
  - $c^* = f(k^*) - k^*$
  - $G(k_t, k_{t+1}) = u(f(k_t) - k_{t+1})$
  - $\lim_{t\to\infty} \beta^t G_1(k_t, k_{t+1})k_t = 0$
  - $\lim_{t\to\infty} \beta^t G_1(k^*, k^*)k^* = 0$

Global Convergence

Assumptions:

- u and f are strictly increasing, strictly concave, and continuously diffrentiable
- $f(0) = 0$, $\lim_{k\to 0} f'(k) = \infty$, and $\lim_{k\to\infty} f'(k) \equiv b < 1$
- $\lim_{c\to 0} u'(c) = \infty$
- $\beta \in (0, 1)$

Problem

$$V(k) \equiv \max_{k' \in [0, f(k)]} \left\{ u\left[f(k) - k'\right] + \beta V(k') \right\}$$

$k' = g(k)$ satisfies the first order condition

$$u'\left[f(k) - k'\right] = \beta V'(k')$$

Properties of g(k)

1. $g(k)$ is single-valued for all k
   - follows from strict concavity of u and V
2. $g(k)$ is strictly increasing
   - Intuition: strict concavity and additivity of the different consumption goods (over time) amounts to assuming that the different goods are normal goods. Specifically, consumption in the future is a normal good. Therefore, a larger initial wealth commands larger savings

Properties of g(k)

Proof.

An application of implicit function theorem

Define

$$H(k,k') = u'\left[f(k) - k'\right] - \beta V'(k') = 0$$

Then

$$\frac{\partial k'}{\partial k} = -\frac{\frac{\partial H(k,k')}{\partial k}}{\frac{\partial H(k,k')}{\partial k'}} = \frac{u''\left[f(k)-k'\right]f'(k)}{u''[f(k)-k']+\beta V''(k')} > 0 \qquad \square$$

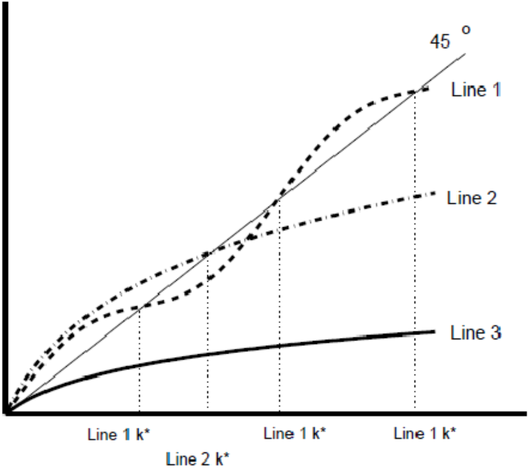
Properties of g(k)

3. $g(k)$ is continuous
4. $g(0) = 0$
5. There exists $\bar{k}$ s.t. $g(k) < \bar{k}$ for all $k < \bar{k}$
   - The first part follows from feasibility: because $c \geq 0$, k' cannot exceed $f(k)$. Our assumptions on f then guarantee that $f(k) < k$ for high enough values of k: the slope of f approaches a number less than 1 as k goes to infinity. So $g(k) < k$ follows

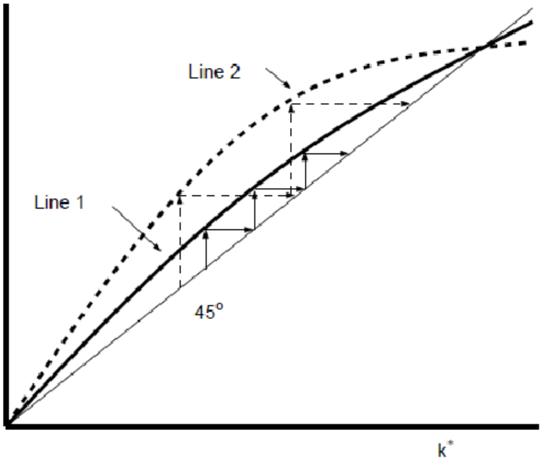
Properties of g(k)

6. Moreover, $\bar{k}$ exceeds $(f')^{-1}(1/\beta) = k^*$
   - $\bar{k}$ must be above the value that maximizes $f(k) - k$, since $f(k)$ is above k for very small values of k and f is strictly concave
   - $\bar{k} > (f')^{-1}(1) > (f')^{-1}(1/\beta)$

Line 2: only one steady state; Concavity rules out line 1; existence rules out line 3



Dynamics: the speed of convergence



Speed of convergence

- There is no simple way to summarize, in a quantitative way, the speed of convergence for a general decision rule
- However, for a limited class of decision rules, it can be measured simply by looking at the slope
    - This is an important case, for it can be used locally to approximate the speed of convergence around the steady state k^*