Advanced Macroeconomics I

Lecture 3 (2)

Zhe Li

SUFE

Spring 2011

Linear rule

- The accumulation path will spend infinite time arbitrarily close to the steady state
- In a very small region a continuous function can be arbitrarily well approximated by a linear function, using the first-order Taylor expansion of the function
    - we will be able to approximate the speed of convergence arbitrarily well as time passes

Linear rule

- If the starting point is far from the steady state, we will make mistakes that might be large initially, but these mistakes will become smaller and smaller and eventually become unimportant

Local and global convergence

- The global convergence theorem applies only for the one-sector growth model
- Local convergence applies to a general dynamic system

Local dynamics

- The first-order Taylor series expansion of the decision rule gives

$$k' = g(k) \approx g(k^*) + g'(k^*)(k - k^*)$$
$$k' - k^* = g'(k^*)(k - k^*)$$

interpret $g'(k^*)$ as a measure of the rate of convergence

  - stationary if $g'(k^*) < 1$
  - If $g'(k^*)$ close to zero, convergence is fast (gap decreases significantly each period)

Linearization for a general dynamic system

- The task is now to find $g'(k^*)$
- Linearize the Euler equation
  - Lead to difference equation in k_t
  - One of the solutions to this difference equation will be the one we are looking for

Local convergence cnditions

Theorem

Let $x_t \in R^n$. *Given* $x_{t+1} = h(x_t)$ *with a stationary point* $x^* : x^* = h(x^*)$ *if*

1. *h is continuously differentiable with Jacobian* $H(x^*)$ *around* x^*
2. $I - H(x^*)$ *is non-singular*

then there is a set of initial conditions x_0, *of dimension equal to the number of eigenvalues of* $H(x^*)$ *that are less than* 1 *in absolute value, for which* $x_t \longrightarrow x^*$

Jacobian matrix

- First derivatives: it changes the value of $x_t - x^*$, but does not change the direction

$$H = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

- $x_t - x^*$ is the eigenvector of H, $H\left(x_t - x^*\right) = \lambda\left(x_t - x^*\right)$, eigenvalues: $|H - \lambda I| = 0$

Converging Dynamics - n=1

Examples

There is only one eigenvalue:

$k' - k^* = g'(k^*)(k - k^*) : \ \lambda = g'(k^*)$

1. $|\lambda| \geq 1$, no initial condition leads to x_t converging to x^*

   In this case, only for $x_0 = x^*$ will the system stay in x^* if $\lambda = 1$

2. $|\lambda| < 1$, $x_t \longrightarrow x^*$ for any value of x_0

Converging Dynamics - n=2

Examples

There are two eigenvalues: $\lambda_1, \ \lambda_2$

$$\begin{aligned} k' - k^* &= g_1(k^*, h^*)\,(k - k^*) \\ &\quad + g_2(k^*, h^*)\,(h - h^*) \\ h' - h^* &= f_1(k^*, h^*)\,(k - k^*) \\ &\quad + f_2(k^*, h^*)\,(h - h^*) \end{aligned}$$

Converging Dynamics - n=2

$$\binom{k'-k^*}{h'-h^*} = \begin{bmatrix} g_1(k^*, h^*) & g_2(k^*, h^*) \\ f_1(k^*, h^*) & f_2(k^*, h^*) \end{bmatrix} \binom{k-k^*}{h-h^*}$$

1. $|\lambda_1|, |\lambda_2| \geq 1$, no initial condition $x_0 = \binom{k_0}{h_0}$ leads to $x_t = \binom{k_t}{h_t}$ converging to $x^* = \binom{k^*}{h^*}$
2. $|\lambda_1| < 1, |\lambda_2| \geq 1$, dimention of $x_0' s$ leading to *convergence* is 1
3. $|\lambda_1| < 1, |\lambda_2| < 1$, dimention of $x_0' s$ leading to *convergence* is 2, $x_t \longrightarrow x^*$ for any value of x_0

Interpret the eigenvalues

- Let the number of eigenvalues less than 1 in absolute value be denoted by m.
  - dimension of the set of initial x_0 leading to x^*
- Interpret m as the degrees of freedom: converge with any initial value x_0
- Let the number of economic restrictions on initial conditions be denoted by $\hat{m}$
  - the restrictions coming from physical conditions in our economic model

Solutions

- An interpretation: we have $\hat{m}$ equations and m unknowns

1. $m = \hat{m}$: there is a unique convergent solution to the difference equation system
2. $m < \hat{m}$: No convergent solution obtains
3. $m > \hat{m}$: There is "indeterminacy", i.e. many solutions

Solving for the speed of convergence

- Derive the Euler equation:
  $F(k_t, k_{t+1}, k_{t+2}) = 0$
  $u'[f(k_t) - k_{t+1}] -$
  $\beta u'[f(k_{t+1}) - k_{t+2}] f'(k_{t+1}) = 0$
- k^* is a steady state $\Leftrightarrow F(k^*, k^*, k^*) = 0$

Linearize the Euler equation

Define $\hat{k}_t = k_t - k^*$, and using first-order Taylor approximation derive a_0, a_1, and a_2 such that

$$a_2\hat{k}_{t+2} + a_1\hat{k}_{t+1} + a_0\hat{k}_t = 0$$

$$\begin{aligned}
&\beta u''(c^*)f'(k^*)\hat{k}_{t+2} \\
&- \left[\begin{array}{c} u''(c^*) + \beta u''(c^*)\left[f'(k^*)\right]^2 \\ +\beta u'(c^*)f''(k^*) \end{array} \right] \hat{k}_{t+1} \\
&+ u''(c^*)f'(k^*)\hat{k}_t = 0
\end{aligned}$$

Difference equation

$$\hat{k}_{t+2} - \left[\begin{array}{c} \frac{1}{\beta f'(k^*)} + f'(k^*) \\ + \frac{u'(c^*) f''(k^*)}{u''(c^*) f'(k^*)} \end{array} \right] \hat{k}_{t+1} + \frac{1}{\beta} \hat{k}_t = 0$$

A first-order system

Write the Euler equation as a first-order system: define $x_t = \binom{\hat{k}_{t+1}}{\hat{k}_t}$ and then

$$x_{t+1} = Hx_t$$

$$\begin{pmatrix} \hat{k}_{t+2} \\ \hat{k}_{t+1} \end{pmatrix} = H \begin{pmatrix} \hat{k}_{t+1} \\ \hat{k}_t \end{pmatrix}$$

A first-order system

$$H = \begin{pmatrix} 1 + \frac{1}{\beta} + \frac{u'(c^*)f''(k^*)}{u''(c^*)f'(k^*)} & -\frac{1}{\beta} \\ 1 & 0 \end{pmatrix}$$

$$\beta f'(k^*) = 1$$

Look for eigenvalues

- From the characteristic polynomial given by $|H - \lambda I| = 0$
- Decompose H as follows:

$$H = V \Lambda V^{-1} \Longrightarrow \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

- λ_1 and λ_2 are (distinct) eigenvalues of H; V is a matrix of eigenvectors of H

A change of variables

- Given $x_{t+1} = Hx_t$, multiply by V^{-1}

$$\begin{aligned} V^{-1}x_{t+1} &= V^{-1}Hx_t = V^{-1}V\Lambda V^{-1}x_t \\ &= \Lambda V^{-1}x_t \end{aligned}$$

- let $z_t \equiv V^{-1}x_t$ *and* $z_{t+1} = V^{-1}x_{t+1}$, where $z_t = \binom{z_{1t}}{z_{2t}}$

$$z_{t+1} = \Lambda z_t = \Lambda^t z_0$$

- So $z_{1t} = z_{10}\lambda_1^t$ and $z_{2t} = z_{20}\lambda_2^t$

Solve k

- We can go back to x_t by premultiplying z_t by V :

$$x_t = Vz_t = V\begin{pmatrix} z_{1t} \\ z_{2t} \end{pmatrix}$$

$$\begin{aligned} x_t &= z_{10}\lambda_1^t \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} + z_{20}\lambda_2^t \begin{pmatrix} V_{12} \\ V_{22} \end{pmatrix} \\ &= \begin{pmatrix} \hat{k}_{t+1} \\ \hat{k}_t \end{pmatrix} \end{aligned}$$

Eigenvalues

$$\begin{vmatrix} 1+\frac{1}{\beta}+\frac{u'(c^*)f''(k^*)}{u''(c^*)f'(k^*)}-\lambda & -\frac{1}{\beta} \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \left[1+\frac{1}{\beta}+\frac{u'(c^*)f''(k^*)}{u''(c^*)f'(k^*)}\right]\lambda + \frac{1}{\beta} = 0$$

Eigenvalues

Let

$$
\begin{aligned}
F(\lambda) = \; & \lambda^2 - \\
& \left[1 + \frac{1}{\beta} + \frac{u'(c^*)f''(k^*)}{u''(c^*)f'(k^*)}\right]\lambda \\
& + \frac{1}{\beta}
\end{aligned}
$$

$F(\lambda)$ is a continuous function of λ

Eigenvalues

$$\left.\begin{array}{c} F(0) = \frac{1}{\beta} > 0 \\ F(1) = -\frac{u'(c^*)f''(k^*)}{u''(c^*)f'(k^*)} < 0 \\ F(\infty) = \infty > 0 \end{array}\right\}$$

$$\Longrightarrow \lambda_1 < 1, \lambda_2 > 1$$

A convergent solution

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{k}_t \end{pmatrix} = z_{10}\lambda_1^t \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} + z_{20}\lambda_2^t \begin{pmatrix} V_{12} \\ V_{22} \end{pmatrix}$$
$$\hat{k}_t = c_1\lambda_1^t + c_2\lambda_2^t$$

A convergent solution to the system requires $c_2 = 0$. The remaining constant, c_1, will be determined from

$$k_0 - k^* \equiv \hat{k}_0 = c_1\lambda_1^0 = c_1$$

Convergent rate

$$k_t - k^* = \lambda_1^t (k_0 - k^*)$$

$$k_{t+1} - k^* = \lambda_1 (k_t - k^*)$$

The eigenvalue λ_1 has a particular meaning: it measures the (inverse of the) rate of convergence to the steady state

Empirical Evidence of Convergence

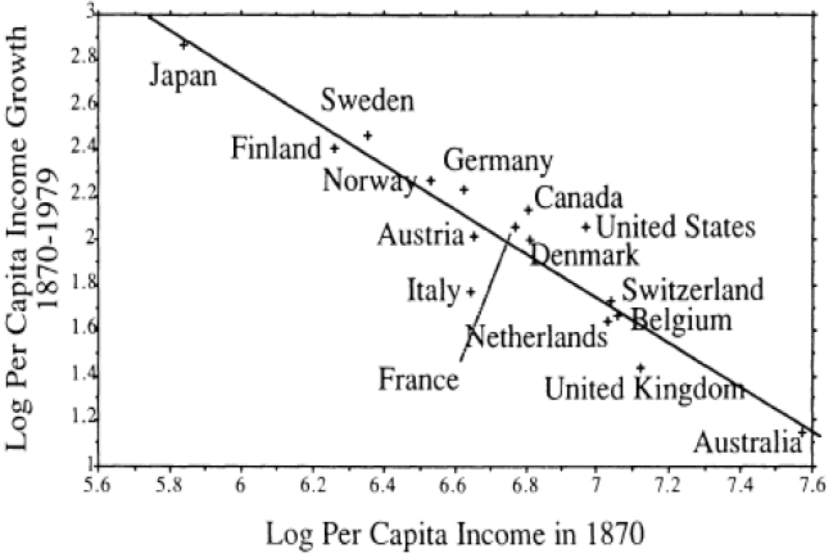
- Whether poor conutries tend to grow faster than rich countries?
  - Baumol (1986) examines convergence from 1870 to 1979 among 16 industralized countries

$$\ln \frac{y_{i,t}}{y_{i,0}} = a + b \ln y_{i,0} + e_i$$

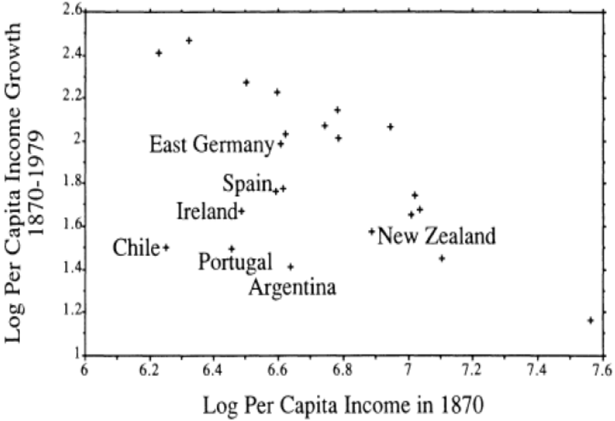
$$\begin{aligned} a &= 8.457 \text{ and } b = -0.995 \\ R^2 &= 0.87 \end{aligned}$$

Interpret Convergence

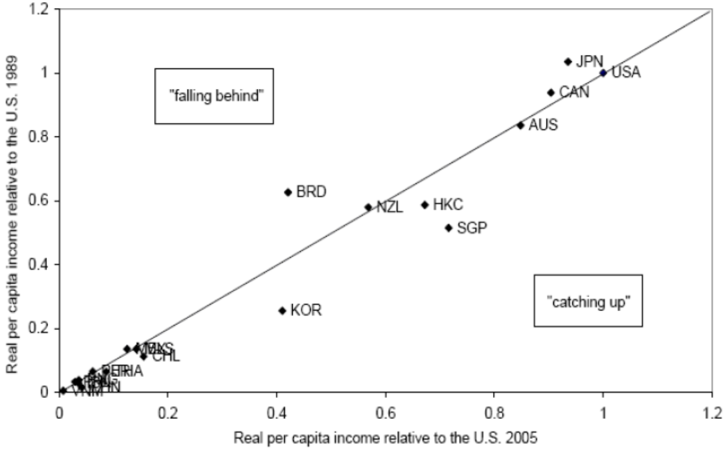
- $b = -1$ corresponds to perfect convergence:
  - higher initial income on average lowers subsequent growth one-for-one
  - and so output per capita in 1979 is uncorrelated with its value in 1870


Log Per Capita Income Growth
1870-1979
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
Japan
Sweden
Finland
Norway
Germany
Canada
United States
Austria
Denmark
Italy
Switzerland
Belgium
Netherlands
France
United Kingdom
Australia
5.6
5.8
6
6.2
6.4
6.6
6.8
7
7.2
7.4
7.6
Log Per Capita Income in 1870

Delong (1988) criticizes: sample selection (only industrialized countries in the sample, while they were poor 100 years ago)



Catching up



"falling behind"
"catching up"
JPN
USA
CAN
AUS
BRD
NZL
HKC
SGP
KOR
CHL
Real per capita income relative to the U.S. 1989
Real per capita income relative to the U.S. 2005
0
0.2
0.4
0.6
0.8
1
1.2