Advanced Macroeconomics I

Lecture 3 (5)

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Solving for the speed of convergence

- Derive the Euler equation: $F(k_t, k_{t+1}, k_{t+2}) = 0$
  $u'\left[f(k_t) - k_{t+1}\right] - \beta u'\left[f(k_{t+1}) - k_{t+2}\right] f'(k_{t+1}) = 0$
- k^* is a steady state $\Leftrightarrow F(k^*, k^*, k^*) = 0$

Linearize the Euler equation

- Define $\hat{k}_t = k_t - k^*$, and using first-order Taylor approximation derive a_0, a_1, and a_2 such that

$$a_2 \hat{k}_{t+2} + a_1 \hat{k}_{t+1} + a_0 \hat{k}_t = 0$$

$$\begin{aligned} & \beta u''(c^*) f'(k^*) \hat{k}_{t+2} \\ & - \left[u''(c^*) + \beta u''(c^*) \left[f'(k^*) \right]^2 + \beta u'(c^*) f''(k^*) \right] \hat{k}_{t+1} \\ & + u''(c^*) f'(k^*) \hat{k}_t \\ = \; & 0 \end{aligned}$$

$$\hat{k}_{t+2} - \left[\frac{1}{\beta f'(k^*)} + f'(k^*) + \frac{u'(c^*) f''(k^*)}{u''(c^*) f'(k^*)} \right] \hat{k}_{t+1} + \frac{1}{\beta} \hat{k}_t = 0$$

A first-order system

- Write the Euler equation as a first-order system: A difference equation of any order can be written as a first order difference equation by using vector notation: Define $x_t = \binom{\hat{k}_{t+1}}{\hat{k}_t}$ and then

$$x_{t+1} = Hx_t$$

$$\begin{pmatrix} \hat{k}_{t+2} \\ \hat{k}_{t+1} \end{pmatrix} = H \begin{pmatrix} \hat{k}_{t+1} \\ \hat{k}_t \end{pmatrix}$$

$$H = \begin{pmatrix} 1 + \frac{1}{\beta} + \frac{u'(c^*)f''(k^*)}{u''(c^*)f'(k^*)} & -\frac{1}{\beta} \\ 1 & 0 \end{pmatrix}$$

$\beta f'(k^*) = 1$

Look for eigenvalues

From the characteristic polynomial given by

$$|H - \lambda I| = 0$$

Using spectral decomposition, we can decompose H as follows:

$$H = V\Lambda V^{-1} \Longrightarrow \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

λ_1 and λ_2 are (distinct) eigenvalues of H, V is a matrix of eigenvectors of H

A change of variables

$$x_{t+1} = Hx_t$$

$$V^{-1}x_{t+1} = V^{-1}Hx_t = V^{-1}V\Lambda V^{-1}x_t = \Lambda V^{-1}x_t$$

let $z_t \equiv V^{-1}x_t, z_{t+1} = V^{-1}x_{t+1}$, $z_t = \binom{z_{1t}}{z_{2t}}$

$$z_{t+1} = \Lambda z_t = \Lambda^t z_0$$

$$\begin{aligned} z_{1t} &= z_{10}\lambda_1^t \\ z_{2t} &= z_{20}\lambda_2^t \end{aligned}$$

Solve k

We can go back to x_t by premultiplying z_t by V :

$$\begin{aligned} x_t &= Vz_t \\ &= V\begin{pmatrix} z_{1t} \\ z_{2t} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} x_t &= z_{10}\lambda_1^t\begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} + z_{20}\lambda_2^t\begin{pmatrix} V_{12} \\ V_{22} \end{pmatrix} \\ &= \begin{pmatrix} \hat{k}_{t+1} \\ \hat{k}_t \end{pmatrix} \end{aligned}$$

Eigenvalues

$$\begin{vmatrix} 1 + \frac{1}{\beta} + \frac{u'(c^*)f''(k^*)}{u''(c^*)f'(k^*)} - \lambda & -\frac{1}{\beta} \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \left[1 + \frac{1}{\beta} + \frac{u'(c^*)f''(k^*)}{u''(c^*)f'(k^*)}\right]\lambda + \frac{1}{\beta} = 0$$

Eigenvalues

Let

$$F(\lambda) = \lambda^2 - \left[1 + \frac{1}{\beta} + \frac{u'(c^*)f''(k^*)}{u''(c^*)f'(k^*)}\right]\lambda + \frac{1}{\beta}$$

$F(\lambda)$ is a continuous function of λ

$$\left.\begin{array}{c} F(0) = \frac{1}{\beta} > 0 \\ F(1) = -\frac{u'(c^*)f''(k^*)}{u''(c^*)f'(k^*)} < 0 \\ F(\infty) = \infty > 0 \end{array}\right\} \Longrightarrow \lambda_1 < 1,\ \lambda_2 > 1$$

A convergent solution

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{k}_t \end{pmatrix} = z_{10}\lambda_1^t \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} + z_{20}\lambda_2^t \begin{pmatrix} V_{12} \\ V_{22} \end{pmatrix}$$

$$\hat{k}_t = c_1\lambda_1^t + c_2\lambda_2^t$$

A convergent solution to the system requires $c_2 = 0$. The remaining constant, c_1, will be determined from

$$k_0 - k^* \equiv \hat{k}_0 = c_1\lambda_1^0 = c_1$$

$$k_t - k^* = \lambda_1^t \left(k_0 - k^*\right)$$

$$k_{t+1} - k^* = \lambda_1 \left(k_t - k^*\right)$$

It can thus be seen that the eigenvalue λ_1 has a particular meaning: it measures the (inverse of the) rate of convergence to the steady state