Advanced Macroeconomics I

Lecture 3 (7)

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Continuous Time

$$\int_{t=0}^{\infty} \ln c(t)\; e^{-\rho t} dt$$

$$\begin{aligned} s.t.\ \dot{k}(t) &= f(k(t)) - c(t) - k(t) \\ f(k(t)) &= Ak^{\alpha}(t) \\ \dot{k}(t) &= \lim_{\Delta \to 0} (k(t+\Delta) - k(t)) = f(k(t)) - c(t) - k(t) \end{aligned}$$

$$\begin{aligned} k_0 &> 0 \text{ given} \\ 0 &< \rho < 1,\ \ [exp(-0.04) = 0.96 = \beta] \end{aligned}$$

Hamiltonian

$$H = e^{-\rho t} \ln c(t) + \lambda(t) \left[f(k(t)) - c(t) - k(t) \right]$$

First Order conditions:

$$\begin{aligned} H_c &= e^{-\rho t} 1/c(t) - \lambda(t) = 0 \\ H_k &= \lambda(t) \left[f'(k(t)) - 1 \right] = -\dot{\lambda}(t) \end{aligned}$$

Transversality condition:

$$\lim_{t \to \infty} \lambda(t) k(t + \Delta) = 0$$

$$
\begin{aligned}
L &= \int_{t=0}^{\infty} \ln c(t)\ e^{-\rho t} dt + \ldots \\
&\quad + \lambda(t-\Delta)\left(f(k(t-\Delta)) - c(t-\Delta) - k(t)\right) \\
&\quad + \lambda(t)\left[f(k(t)) - c(t) - k(t+\Delta)\right] + \ldots
\end{aligned}
$$

$\lambda(t-\Delta) : (k(t) - k(t-\Delta)) = f(k(t-\Delta)) - c(t-\Delta) - k(t-\Delta)$
$\lambda(t) : (k(t+\Delta) - k(t)) = f(k(t)) - c(t) - k(t)$

FOC with respect to $k(t)$:

$$
-\lambda(t-\Delta) + \lambda(t) f'(k(t)) = 0
$$

$$
\lambda(t)\left(f'(k(t)) - 1\right) = \lambda(t-\Delta) - \lambda(t)
$$

$$
H_k = \lambda(t)\left(f'(k(t)) - 1\right) = -\dot{\lambda}(t)
$$

$$
\begin{aligned}
H_c &= e^{-\rho t} 1/c(t) - \lambda(t) = 0 \\
-\rho t - \ln c(t) &= \ln \lambda(t) \\
-\rho - \frac{\dot{c}(t)}{c(t)} &= \frac{\dot{\lambda}(t)}{\lambda(t)}
\end{aligned}
$$

$$
\begin{aligned}
H_k &= \lambda(t) \left[f'(k(t)) - 1 \right] = -\dot{\lambda}(t) \\
\frac{\dot{\lambda}(t)}{\lambda(t)} &= -\left[f'(k(t)) - 1 \right]
\end{aligned}
$$

Steady State

define steady state $\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = 0$

$$\begin{aligned} f'(k^*) - 1 &= \rho \\ k^* &= f'^{-1}(1+\rho) \end{aligned}$$

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - c(t) - k(t) \\ c^* &= f(k^*) - k^* \end{aligned}$$

Taylor expansion around steady state

$$\dot{c}(t) = c(t)\left[f'(k(t)) - 1 - \rho\right]$$

$$\begin{aligned}\dot{c}(t) &= c^*\left[f'(k^*) - 1 - \rho\right] + \left[f'(k^*) - 1 - \rho\right](c(t) - c^*) \\ &\quad + c^* f''(k^*)(k(t) - k^*)\end{aligned}$$

$$\dot{k}(t) = f(k(t)) - c(t) - k(t)$$

$$\dot{k}(t) = \left[f'(k^*) - 1\right](k(t) - k^*) - (c(t) - c^*)$$

$$\begin{pmatrix}\dot{c}(t) \\ \dot{k}(t)\end{pmatrix} = \begin{bmatrix} 0 & c^* f''(k^*) \\ -1 & \rho \end{bmatrix}\begin{pmatrix}c(t) - c^* \\ k(t) - k^*\end{pmatrix}$$