Advanced Macroeconomics I

Lecture 4 (2)

Zhe Li

SUFE

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Neoclassical growth model with sequential trade

- The following are the prices involved in this market structure:
  - Price of capital services at t: R_t
  - $R_t = r_t + 1 - \delta$
- Price of labor:
  - w_t price of labor at t relative to (in terms of) consumption goods at t

Equilibrium with sequential trade

Definition

A competitive equilibrium is a sequence $\left\{R_t^*, w_t^*, c_t^*, K_{t+1}^*, n_t^*\right\}_{t=0}^{\infty}$ such that
1.)

$$\begin{aligned}
&\left\{c_t^*, K_{t+1}^*, n_t^*\right\}_{t=0}^{\infty} \\
= \; &\arg \max_{\left\{c_t, K_{t+1}, n_t\right\}_{t=0}^{\infty}} \left\{\sum_{t=0}^{\infty} \beta^t u(c_t)\right\}
\end{aligned}$$

Definition

s.t.

$$c_t + K_{t+1} = R_t^* K_t + n_t w_t^*$$

k_0 given and a no-Ponzi-game condition

2.) $\left\{K_{t+1}^*, n_t^*\right\}_{t=0}^{\infty}$ solves the firms' problem:

$$\begin{aligned} & \forall t \ \left(K_t^*, 1\right) \\ = & \arg\max_{K_t, n_t} \left\{ \begin{array}{c} F\left(K_t, n_t\right) - R_t^* K_t \\ + \left(1 - \delta\right) K_t - w_t^* n_t \end{array} \right\} \end{aligned}$$

Definitions

3.) Market clearing (feasibility):

$$\forall t \; c_t^* + K_{t+1}^* = F\left(K_t^*, 1\right) + \left(1 - \delta\right) K_t^*$$

Consumer's problem

- A sequence of Lagrange Multiplier
  FOC

$$
\begin{aligned}
c_t &: \beta^t u'(c_t^*) = \beta^t \lambda_t^* \\
c_{t+1} &: \beta^{t+1} u'(c_{t+1}^*) = \beta^{t+1} \lambda_{t+1}^*
\end{aligned}
$$

$$K_{t+1} : \beta^t \lambda_t^* = \beta^{t+1} R_{t+1}^* \lambda_{t+1}^*$$

Optimal conditions

- Marginal rate of substitution = interest rate

$$\frac{\lambda_t^*}{\lambda_{t+1}^*} = \beta R_{t+1}^*$$

$$\frac{u'\left(c_t^*\right)}{u'\left(c_{t+1}^*\right)} = \beta R_{t+1}^*$$

- Firm's optimal:

$$R_t^* = F_k\left(K_t^*, 1\right) + 1 - \delta$$

Equilibrium

- Euler equation

$$u'(c_t^*) = \beta u'(c_{t+1}^*) \left(F_k\left(K_t^*, 1\right) + 1 - \delta\right)$$

- This is identical to the planner's Euler equation
- The sequential market equilibrium is the same as the Arrow-Debreu-McKenzie date-0 equilibrium
- Both are Pareto-optimal

Recursive competitive equilibrium

- Treating all maximization problems as split into decisions concerning **today** versus **the entire future**
- Instead of having sequences, a recursive competitive equilibrium is a set of functions
  - quantities, utility levels, and prices, as functions of the "state"
  - These functions allow us to say what will happen in the economy for every specific consumer, given an arbitrary choice of the initial state

- Assume time endowment $= 1$, leisure is not valued
- Recall the central planner's problem:

$$V(K) = \max_{c, K' \geq 0} \{u(c) + \beta V(K')\}$$

$$s.t.\ c + K' = F(K, 1) + (1 - \delta) K$$

The neoclassical growth model

Decentralized recursive economy

- The individual's budget constraint will no longer be expressed in terms of physical units, but in terms of sources and uses of funds at the going market prices, $\{R_t, w_t\}_{t=0}^{\infty}$ with the equilibrium levels given by

$$
\begin{aligned}
R_t^* &= F_K(K_t^*, 1) + 1 - \delta \\
w_t^* &= F_n(K_t^*, 1)
\end{aligned}
$$

Aggregate states

- If $\bar{K}$ denotes the (current) aggregate capital stock, then

$$R = R(\bar{K}), \; w = w(\bar{K})$$

Individual states

- For one individual:

$$c + K' = R(\bar{K})K + w(\bar{K})$$

  two variables are key to the agent: his own level of capital, K, and the aggregate level of capital, $\bar{K}$, which will determine his income

- Value function

$$V(K, \bar{K}) = \max_{c, K' \geq 0} \{u(c) + \beta V(K', \bar{K}')\}$$

Law of motion-aggregate capital

- The agent's perceived law of motion of aggregate capital
  - We assume that he will perceive this law of motion as a function of the aggregate level of capital
  - His perception is rational - it will correctly correspond to the actual law of motion:

$$\bar{K}' = G(\bar{K})$$

where G is a result of the representative agent's equilibrium capital accumulation decisions

Dynamic problem in decentralized economy

- The consumer's complete dynamic problem in the decentralized economy:

$$V(K, \bar{K}) = \max_{c, K' \geq 0} \{u(c) + \beta V(K', \bar{K}')\}$$

s.t.

$$c + K' = R(\bar{K})K + w(\bar{K})$$

$$\bar{K}' = G(\bar{K})$$

Policy function

- A policy function for the individual's law of motion for capital:

$$\begin{aligned} K' &= g(K, \bar{K}) \\ &= \arg \max_{K' \in [0, R(\bar{K})K + w(\bar{K})]} \left\{ \begin{array}{c} u(R(\bar{K})K + w(\bar{K}) - K') \\ +\beta V(K', \bar{K}') \end{array} \right\} \end{aligned}$$

- Aggregate policy function

$$\bar{K}' = G(\bar{K})$$

A recursive competitive equilibrium

Definition

A recursive competitive equilibrium is a set of functions:

Quantities $G(\bar{K})$, $g(K, \bar{K})$,

Lifetime utility level $V(K, \bar{K})$,

and Prices $R(\bar{K})$, $w(\bar{K})$ such that

1. $V(K, \bar{K})$ solves the consumer's problem, and $g(K, \bar{K})$ is the associated policy function

A recursive competitive equilibrium

Definition

2 Prices are competitively determined:

$$\begin{aligned} R(\bar{K}) &= F_K(\bar{K}, 1) + 1 - \delta \\ w(\bar{K}) &= F_n(\bar{K}, 1) \end{aligned}$$

In the recursive formulation, prices are stationary functions, rather than sequences

A recursive competitive equilibrium

Definition

3 Individual and aggregate consistency:

$$G(\bar{K}) = g(\bar{K}, \bar{K}) \ \forall \bar{K}$$

- The third condition — consistency is the distinctive feature of the recursive formulation of competitive equilibrium

A distinctive feature

- Whenever the individual consumer is endowed with a level of capital equal to the aggregate level, his own individual behavior will exactly mimic the aggregate behavior
  - for example, only one single agent in the economy owns all the capital, or there is a measure one of agents
- The aggregate law of motion perceived by the agent must be consistent with the actual behavior of individuals

Market Clears?

- $$c + \bar{K}' = F(\bar{K}, 1) + (1 - \delta)\,\bar{K}$$

Theorem

Euler Theorem: if the production technology exhibits constant returns to scale (that is, if the production function is homogeneous of degree 1), then

$$F(\bar{K}, 1) + (1 - \delta)\,\bar{K} = R(\bar{K})\bar{K} + w(\bar{K})$$

Off-equilibrium

- Solving for the functions V and g, which specify "off-equilibrium" behavior: what the agent would do if he were different from the representative agent?
  - In order to justify the equilibrium behavior we need to see that the postulated, chosen path, is not worse than any other path
  - $V(K, \bar{K})$ precisely allows you to evaluate the futurec onsequences for these behavioral alternatives, thought of as one-period deviations

Off-equilibrium

- Implicitly, in the sequential approach, we check off-equilibrium behavior
    - although in that approach one typically simply derives the first-order (Euler) equation and imposes $K = \bar{K}$ there
    - Knowing that the F.O.C. is sufficient, one does not need to look explicitly at alternatives

Optimal?

- $$V(K, \bar{K}) = \max_{c, K' \geq 0} \{u(c) + \beta V(K', \bar{K}')\}$$

  s.t.

  $$c + K' = R(\bar{K})K + w(\bar{K})$$

  $$\bar{K}' = G(\bar{K})$$

Optimal?

- $$V(K,\bar{K}) = \max_{K' \geq 0} \left\{ \begin{array}{c} u(R(\bar{K})K + w(\bar{K}) - K') \\ +\beta V(K', \bar{K}') \end{array} \right.$$

- F.O.C:

$$\begin{aligned} & u'(R(\bar{K})K + w(\bar{K}) - g(K,\bar{K})) \\ = \ & \beta V'(K', \bar{K}') \end{aligned}$$

$g(K,\bar{K})$ individual policy function

Optimal?

- Envelope condition:

$$V(K,\bar{K}) = \max_{K'\geq 0} \left\{ \begin{array}{c} u(R(\bar{K})K + w(\bar{K}) \\ -K') \\ +\beta V(K',\bar{K}') \end{array} \right\}$$

$$\begin{aligned} V'(K,\bar{K}) &= R(\bar{K})u'(R(\bar{K})K \\ &\quad +w(\bar{K}) - g(K,\bar{K})) \end{aligned}$$

Optimal?

- Euler equation:

$$
\begin{aligned}
& u'(R(\bar{K})K + w(\bar{K}) - g(K, \bar{K})) \\
= \; & \beta R(G(\bar{K})) \\
& u' \begin{bmatrix} R(G(\bar{K}))g(K, \bar{K}) \\ +w(G(\bar{K})) - G(G(K)) \end{bmatrix}
\end{aligned}
$$

Optimal!

- Using Euler theorem and consistency:

$$
\begin{aligned}
& u'\left(F(\bar{K},1) + (1-\delta)\,\bar{K} - G(\bar{K})\right) \\
= \;& \beta u' \left[\begin{array}{c} F(G(\bar{K}),1) + (1-\delta)\,G(\bar{K}) \\ -G(G(\bar{K})) \end{array} \right] \\
& \times \left[F_1(G(\bar{K})) + 1 - \delta\right]
\end{aligned}
$$