Advanced Macroeconomics I

Lecture 5 (1)

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Uncertainty

- Aggregate productivity

$$Y_t = x_t k_t^{\alpha} l_t^{1-\alpha}$$

- Aggregate productivity is often uncertain, but seriel correlated
    - Markov Chain
    - Linear stochastic difference equations

Markov Chain

Definition

Let $x_t \in X$, where $X = \{\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n\}$ is a finite set of values. A stationary Markov Chain is a stochastic process $\{x_t\}_{t=1}^{\infty}$ defined by a set X, a transition matrix, P, and an initial probability distribution, π_0 for x_0.

Markov Chain

- P probabilities:
  $P_{ij} = \Pr\left\{x_{t+1} = \tilde{x}_j \mid x_t = \tilde{x}_i\right\}$
- Given π_0, π_1 is the probability distribution of x_1 as of time $t = 0$, $\pi_1 = \pi_0 P$

Stationary distribution

Definition

A stationary (or invariant) distribution for P is a probability vector π such that $\pi = \pi P$

A stationary distribution satisfies $\pi I = \pi P$,

$$\pi \left[I - P \right] = 0$$

π is an eigenvector of P, associated with the eigenvalue $\lambda = 1$

Asymptotically stationary

- If π_t convergences to a **number** π_∞ as $t \longrightarrow \infty$

$$\pi_\infty = \pi_\infty P$$

- If π_∞ does not depend on the initial π_0
- then the stochastic process is said to be "asymptotically stationary", with a unique invariant distribution

Linear stochastic difference equations

- Auto-regressive process $AR(1)$

$$y_{t+1} = b + \rho y_t + \varepsilon_{t+1}$$

- Assume

$$\begin{aligned} E_t\left[\varepsilon_{t+1}\right] &= 0 \\ E_t\left[\varepsilon_{t+1}^2\right] &= \sigma^2 \\ E_t\left[\varepsilon_{t+k}\ \varepsilon_{t+k+1}\right] &= 0 \end{aligned}$$

Asymptotically stationary

The process $\{y_t\}_{t=0}^{\infty}$ is not stationary in general, but it is Asymptotically stationary as $t \to \infty$

$$E_0(y_t) = \rho^t y_0 + \frac{b}{1-\rho}(1-\rho^t)$$

if $|\rho| < 1$, as $t \to \infty$,

$$E_0(y_t) = \frac{b}{1-\rho}$$

Stochastic neoclassical growth model

- Introduce uncertainty into the neoclassical growth model:

$$F_t(k_t, 1) = x_t f(k_t), \; x_t \in X$$

- The sequence of productivity

$$x^t = (x_t, x_{t-1}, ..., x_0)$$

Expected life-time utility

- Expected life-time utility

$$\sum_{t=0}^{\infty} \sum_{x^t \in X^t} \beta^t \pi(x^t) u\left[c_t(x^t)\right]$$

$$\equiv \; E\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right]$$

  - $u\left[c_t(x^t)\right]$: utility in t if $x^t = (x_t, x_{t-1}, ..., x_0)$ realized
  - $\pi(x^t)$: probability of occurrence of the event $(x_t, x_{t-1}, ..., x_0)$

Planning problem in sequential form

- Consumers' problem

$$\max_{\{c_t(x^t),\ k_{t+1}(x^t)\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right]$$

$$\begin{aligned} & s.t.\ c_t(x^t) + k_{t+1}(x^t) \\ = \ & x_t f\left(k_t(x^{t-1})\right) + (1-\delta) k_t(x^{t-1}) \\ & \forall \left(t, x^t\right), \quad k_0 \text{ given} \end{aligned}$$

Stochastic Euler equation

- Foc. with k_{t+1} :

$$\begin{aligned} & \pi(x^t)u'\left[c_t(x^t)\right] \\ = & \sum_{x^{t+1}\in X^{t+1}} \beta\pi(x^{t+1})u'\left[c_{t+1}(x^{t+1})\right] \\ & \times \left\{x_{t+1}f'(k_{t+1}(x^t)) + 1 - \delta\right\} \end{aligned}$$

Stochastic Euler equation

- denote $\pi\left[\left(x_{t+1}, x^t\right)|x^t\right] \equiv \frac{\pi\left(x_{t+1}, x^t\right)}{\pi\left(x^t\right)}$

$$u'\left[c_t\left(x^t\right)\right] = \sum_{x^{t+1}\in X^{t+1}} \beta\pi\left[\left(x_{t+1}, x^t\right)|x^t\right] u'\left[c_{t+1}\left(x_{t+1}, x^t\right)\right] \times \left\{x_{t+1} f'\left(k_{t+1}\left(x^t\right)\right) + 1 - \delta\right\}$$

Stochastic Euler equation

- Stochastic Euler equation

$$u'\left[c_t(x^t)\right] = E_{x^t}\left\{u'\left[c_{t+1}\left(x_{t+1}, x^t\right)\right] R_{t+1}\right\}$$

$R_{t+1} \equiv x_{t+1} f'(k_{t+1}(x^t)) + 1 - \delta$ is the marginal return on capital realized for each x_{t+1}

Planner's problem: recursive version

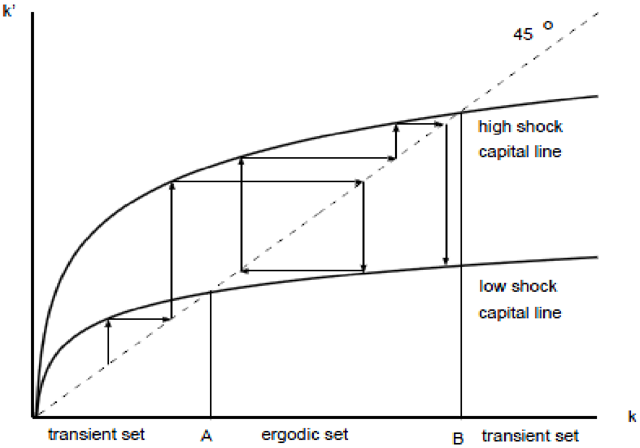
- Value function with x as a state variable

$$V(k,x) = \max_{k'} \left\{ \begin{array}{c} u\left[xf(k) + (1-\delta)k - k'\right] \\ +\beta \sum_{x' \in X} \pi(x'|x) V(k',x') \end{array} \right\}$$

- Solution involves x, $k' = g(k,x)$

Example

$x \in \{x_h, x_l\}$



Transient set

- Transient set, a set of values of capital, which cannot occur in the long run
  - on the left of A: $k_t < OA$
    - The probability of leaving the transient set is equal to the probability of capital reaching a value higher or equal to A
    - which is possible only with a high shock
    - This probability is non-zero and the capital will therefore get beyond A at least once in the long run
  - on the right of B

Ergodic set

- Ergodic set, which is a set, that the capital will never leave once it is there
  - The interval between A and B
  - there is no value of capital from this interval and a shock which would cause the capital to take a value outside of this interval