Advanced Macroeconomics I

Lecture 6 Overlapping-generations model

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Spring 2011

Welfare in models with multiple agents

- New agents are born into the economy each period
  - these individuals' life spans are shorter than the economy's time horizon
- Inter-generation game
  - motive of bequest: why agents would "die" leaving non-consumed savings behind
  - Policy: tax or subsidy on environment, pension

Overlapping generations

- Initial old, meaure 1
- In period 1, and onward, new generations are born, with mearuse 1
- At the end of period 1, the initial old die; every other generation lives for two periods: young and old
- Consumptions: initial old in period 1: c_1^o
- Total consumption in period t: $C_t = c_t^y + c_t^o$

Feasibility

Neoclassical growth model

- Endowment at t, L_t
- a production technology: $Y_t = F(K_t, L_t)$
- feasibility:

$$C_t + K_{t+1} \leq Y_t + (1 - \delta) K_t$$

Efficiency across periods

Definition

A feasible consumption allocation c is efficient if there is no alternative feasible allocation $\hat{c}$ such that

$$\begin{aligned} \hat{c}_t &\geq c_t, \forall t \\ \text{and } \hat{c}_t &> c_t, \text{ for some } t \end{aligned}$$

Efficiency across agents

Definition

A feasible consumption allocation c_A is Pareto superior to c_B (or c_A "Pareto dominates" c_B) if

1. No agent strictly prefers the consumption path specified by c_B to that specified by c_A
2. At least one agent strictly prefers the allocation c_A to c_B

Welfare improvement

- A feasible consumption allocation c_A is Pareto superior to c_B
  - Whenever c_B is implemented, the existence of c_A implies that a welfare improvement is feasible by modifying the allocation

Pareto optimal

Definition

A consumption allocation is Pareto optimal if

1. It is feasible
2. There is no other feasible allocation $\hat{c} \neq c$ that Pareto dominates c

Accumlation of capital

- You may suspect that the fact that agents' life spans are shorter than the economy's horizon might lead to a different level of capital accumulation than if agents lived forever

Over-accumlation of capital

- Economies in which generations overlap lead to an over-accumulation of capital
  - Inefficiency
  - since an over-accumulation of capital implies that the same consumption pattern could have been achieved with less capital investment
  - hence more goods could have been "freed-up" to be consumed

Under-accumlation of capital

- Economies in which generations overlap lead to an under-accumulation of capital
  - Efficient
  - but not Golden rule

Peter Diamond (1965)

Reference: Diamond, Peter, 1965. "National debt in a neoclassical growth model." American Economic Review 55, p. 1126-1150.

- Agents (in period t):

	size
young	$L_t = L_0(1+n)^t$
old	L_{t-1}

	Labor	Capital	Utility
young	1	0	$u(c_t^y, c_{t+1}^o)$
old	0	k_{t-1}^s	$u(c_{t-1}^y, c_t^o)$

Production technology

- Technology: homogeneous of degree 1
  $F(K, L) = LF\left(\frac{K}{L}, 1\right)$

$$F(K, L) = Lf(k)$$

where $k = \frac{K}{L}$

Social Optimum

- Pareto optimality: take care of all the generations
- Maximize steady state utility: c^y, c^o, for all generations t

$$\max_{c^y,\ c^o} u\left(c^y, c^o\right)$$

Resource constraint:

$$L_t(c^y + k^s_{t+1}) + L_{t-1}c^o \leq F(K^d_t, L_t) + L_{t-1}k^s_t$$

Resource constraint

$$L_t(c^y + k^s_{t+1}) + L_{t-1}c^o \leq F(K^d_t, L_t) + L_{t-1}k^s_t$$

Capital supply = demand

$$L_{t-1}k^s_t = K^d_t = L_t k^d_t$$

$$k^s_t = (1+n)k^d_t$$

$$\text{let } k = k^d_t$$

$$c^y + k^s_{t+1} + \frac{c^o}{1+n} \leq F(\frac{K^d_t}{L_t}, 1) + \frac{k^s_t}{1+n}$$

Golden Rule

- Resource constraint

$$c^y + (1+n)k + \frac{c^o}{1+n} \leq f(k) + k$$

$$c^y + \frac{c^o}{1+n} \leq f(k) - nk$$

- optimal k

$$\max_k f(k) - nk$$

$$f'(k) = n$$

Optimal consumption

- Problem

$$\max_{c^y,\ c^o} u\left(c^y, c^o\right)$$

$$c^y + \frac{c^o}{1+n} \leq f\left(k\right) - nk$$

- Optimal condition

$$\frac{u_1}{u_2} = 1 + n$$

$n \uparrow \implies$ motivate to transfer some consumption to old

Competitive Economy

Fact

Incomplete market: a market is missing, agents cannot trade with future generations

A young agent's problem:

$$\max_{c_t^y,\ c_t^o} u\left(c_t^y,\ c_{t+1}^o\right)$$

$$\text{s.t. } c_t^y + k_{t+1}^s \leq w_t \times 1$$

$$c_{t+1}^o \leq \left(1 + r_{t+1}\right) k_{t+1}^s$$

Competitive Economy

Intertemporal budget constraint:

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} \leq w_t$$

Solution:

$$\frac{u_1}{u_2} = 1 + r_{t+1}$$

Competitive economy – Social optimal?

- Competitive:

$$\frac{u_1}{u_2} = 1 + r_{t+1}$$

- Social optimal:

$$\frac{u_1}{u_2} = 1 + n$$

Agents' problem

- First order condistions give

$$\frac{u_1\left(w_t - k^s_{t+1}, (1 + r_{t+1})k^s_{t+1}\right)}{u_2\left(w_t - k^s_{t+1}, (1 + r_{t+1})k^s_{t+1}\right)} = 1 + r_{t+1}$$

- Taken prices, w_t, r_{t+1}, as given, there is one variable to solve, the saving (supply function of capital) $k^s_{t+1} = s(w_t, r_{t+1})$

Effect of wage income

- Assume that the marginal propensity to save satisfies

$$0 < \frac{\partial s}{\partial w} < 1$$

$$c_t^y + k_{t+1}^s = w_t$$

Effect of interest rate

- The effect of interest rate

$$\frac{\partial s}{\partial r_{t+1}} \underset{?}{\overset{<}{>}} 0$$

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t$$

- price of c_t^y : 1; price of c_{t+1}^o : $\frac{1}{1+r_{t+1}}$

Effect of interest rate

- If $r_{t+1} \uparrow$
  - Income effect: boundle of the goods become less expensive $\longrightarrow c_t^y \uparrow \longrightarrow k_{t+1}^s \downarrow$
  - Substitution effect: relative price of today's goods to tomorrow goods $\uparrow \longrightarrow c_t^y \downarrow \longrightarrow k_{t+1}^s \uparrow$

Firm's problem

Assume 0 depreciation

$$\max_{K_t^d, L_t^d} F(K_t^d, L_t^d) - w_t L_t^d - (1 + r_t) K_t^d + K_t^d$$

$$\begin{aligned} r_t &= F_K(K_t^d, L_t^d) = f'(k_t) \\ w_t &= F_L(K_t^d, L_t^d) = f(k_t) - k_t f'(k_t) \end{aligned}$$

$$k_t \equiv \frac{K_t^d}{L_t^d}, \; f(k) \equiv F(\frac{K}{L}, 1)$$

Market clearing

- Capital: $K_t^d = k_t^s L_{t-1}$
- Labor: $L_t^d = 1 \times L_t$
- Goods:
  $L_t(c_t^y + k_{t+1}^s) + L_{t-1}c_t^o = F(K_t^d, L_t^d) + L_{t-1}k_t^s$

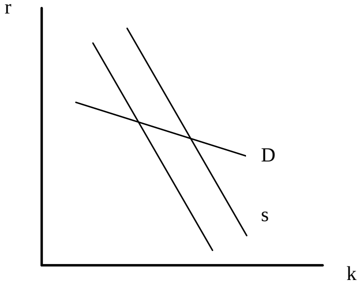
Capital market

- Demand curve: $r_t = f'(k_t)$
- supply curve: $k_t = \frac{s(w_{t-1}, r_t)}{1+n}$

Capital market - Normal case

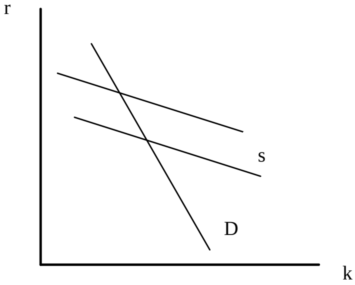
$$w_{t-1} \uparrow \Longrightarrow s \text{ curve} \uparrow \Longrightarrow r_t \downarrow \Longrightarrow \frac{\partial k_t^s}{\partial w_{t-1}} > 0$$

s curve more steeply negatively sloped (or s curve positively sloped)



Capital market - Abnormal case

$$w_{t-1} \uparrow \Longrightarrow s \text{ curve} \uparrow \Longrightarrow r_t \uparrow \Longrightarrow \frac{\partial k_t^s}{\partial w_{t-1}} < 0$$



Income and interest rate

Data: income $\uparrow \Longrightarrow k_t$

- Given r_t, for an individual $\frac{\partial s}{\partial w_{t-1}} > 0$
- $\left(\substack{k_t = \frac{s(w_{t-1}, r_t)}{1+n} \\ r_t = f'(k_t)}\right) \Longrightarrow r_t = f'\left(\frac{s(w_{t-1}, r_t)}{1+n}\right) \Longrightarrow$
  $r_t = \psi(w_{t-1})$
- Normal case: $\psi'(w_{t-1}) < 0$

Labor market

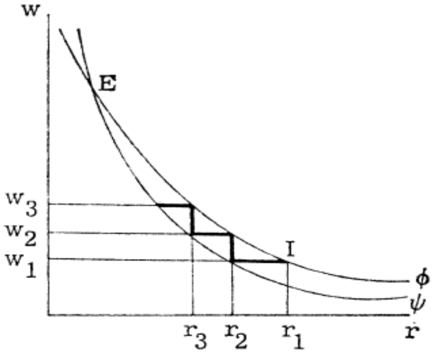
$$w_t = \left[f\left(k_t\right) - k_t f'\left(k_t\right)\right]_{k_t = f'^{-1}(r_t)} \equiv \phi(r_t)$$

$$\phi'(r_t) = -k_t < 0$$

$$r \uparrow \implies k^d \downarrow \implies w_t \downarrow$$

Competitive equilibrium

$w_t = \phi(r_t) \quad r_t = \psi(w_{t-1}) \Longrightarrow w_{t-1} = \psi^{-1}(r_t)$



Stable

slope of ϕ : ϕ'
slope of ψ^{-1} : $\frac{1}{\psi'}$

$$\text{stable if} -\phi' < -\frac{1}{\psi'}$$

$$\phi'\psi' < 1$$

Example

- Utility

$$u(c^y, c^o) = \beta \ln c^y + (1 - \beta) \ln c^o \quad \beta \in (0, 1)$$

- Production function

$$f(k) = Ak^\alpha \quad \alpha \in (0, 1)$$

- Solution: income and substitution effect of interest rate is washed out with lograthmic utility function

Solution

- Saving is proportional to income

$$s(w, r) = (1 - \beta) w$$

- Capital market

$$r_t = \psi(w_{t-1}) = \alpha A \left(\frac{(1 - \beta) w_{t-1}}{1 + n} \right)^{\alpha - 1}$$

- Labor market

$$w_t = \phi(r_t) = (1 - \alpha) \alpha^{\frac{\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} r_t^{\frac{\alpha}{1-\alpha}}$$

Stable

- Stable condition:

$$0 < \phi' \psi' = \alpha < 1$$

- Wage dynamics

$$\begin{aligned} w_t &= \phi\left(\psi\left(w_{t-1}\right)\right) \\ &= \left(1-\alpha\right) A\left(\frac{\left(1-\beta\right) w_{t-1}}{1+n}\right)^{\alpha} \end{aligned}$$

Stable

- Steady state wage

$$w^* = \left[(1-\alpha)A\right]^{\frac{1}{1-\alpha}} \left(\frac{1-\beta}{1+n}\right)^{\frac{\alpha}{\alpha-1}}$$

- Steady state interest rate

$$r^* = \phi^{-1}(w^*) = \frac{\alpha}{1-\alpha}\left(\frac{1+n}{1-\beta}\right) \neq n$$

Competitive equilibrium and Golden rule

- Competitive quilibrium
  $f'(k^*) = r^* = \frac{\alpha}{1-\alpha}\frac{1+n}{1-\beta}$
- Golden rule $f'(k^*) = n$
- $\alpha = 0.36, \beta = 0.6, n = 0.02$
  $r^* = \frac{\alpha}{1-\alpha}\frac{1+n}{1-\beta} = \frac{0.36}{(0.36-1)(0.6-1)}(0.02+1)$
  $= 1.4344$

Missing marekt and Government debt

- If $r > n$, accumulate too little capital
- Missing market across generations
- Enforcement of market: government debt
  - Assume same interest rate (market rate = government rate)
  - Return of debt $(1 + r_t)L_t b$
  - New debt $L_{t+1} b$

Government tax

- Tax on the young

$$(1+r_t)L_t b = L_{t+1}b + T_t$$

$$T_t = L_t b(r_t - n)$$

- Debt holding of per young agent

$$\frac{L_{t+1}b}{L_t} = (1+n)b$$

Budget constraint

- Young agent born at t

$$c_t^y + k_{t+1}^s + (1+n)b \leq w_t - \frac{T_t}{L_t}$$

- Consumption when they become old

$$c_{t+1}^o = (1 + r_{t+1}) \left[k_{t+1}^s + (1+n)b \right]$$

- Intertemporal budget constraint

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t - (r_t - n)b$$

Problem

- Objective:

$$\max_{c_t^y,\ c_{t+1}^o} u\left(c_t^y,\ c_{t+1}^o\right)$$

- Budget constraint:

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t - \left(r_t - n\right) b$$

Solution

- Foc.

$$1 + r_{t+1} = \frac{u_1(w_t - (r_t - n)b - s_t, (1 + r_{t+1})s_t)}{u_2(w_t - (r_t - n)b - s_t, (1 + r_{t+1})s_t)}$$

where $s_t = s(w_t - (r_t - n)b, r_{t+1})$,
$0 < s_w < 1$, and $s_r < 0$

The effect of debt

- Assumbe b can be used as capital
- Focus on normal case $\psi'\phi' < 1$
- If $r^* > n$, can debt move r^* toward n?($\frac{\partial r^*}{\partial b} < 0$?)

Factor markets with debt

- $\begin{pmatrix} k_t = \frac{s(w_{t-1} - (r_{t-1} - n) b, r_t)}{1+n} \\ r_t = f'(k_t) \end{pmatrix}$
- $r_t = f'(\frac{s(w_{t-1} - (r_{t-1} - n) b, r_t)}{1+n})$

$$r_t = \psi(w_{t-1} - (r_{t-1} - n) b)$$

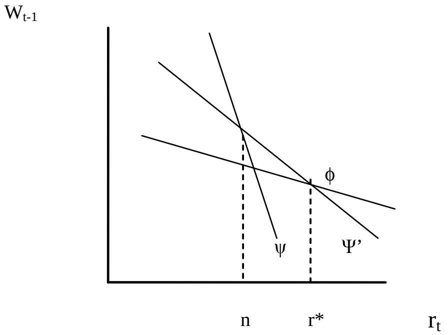
-

$$\begin{aligned} w_t &= [f(k_t) - k_t f'(k_t)]_{k_t = f'^{-1}(r_t)} \\ &\equiv \phi(r_t) \end{aligned}$$

The effect of debt

$$r_t = \psi(w_{t-1} - (r_{t-1} - n)b) \quad w_t = \phi(r_t)$$

Arround the steady state, taken r_{t-1} as given. If $r_{t-1} > n$, debt increases the gap $r - n$



The welfare effect of debt

- Steady state utility $u^* = u(c^y, c^o)$
  - $c^y = w - (r-n)b - s$
  - $c^o = (1+r)s$
- Foc. $s: u_1 = u_2(1+r)$

$$\frac{du^*}{db} = u_1 \left[\begin{array}{c} \frac{dw}{db} - (r-n) \\ -b\frac{dr}{db} - \frac{ds}{db} \end{array} \right] + u_2 \left[\begin{array}{c} (1+r)\frac{ds}{db} \\ +s\frac{dr}{db} \end{array} \right]$$

The welfare effect of debt

- $$\frac{du^*}{db} = u_1 \left[\frac{dw}{db} - (r - n) \right] + [u_2 s - u_1 b] \frac{dr}{db}$$
- *disposible income*
- *interest rate*

The welfare effect of debt

$$\begin{aligned}
\frac{dw}{db} &= \frac{d\left[f\left(k_t\right) - k_t f'\left(k_t\right)\right]}{db} \\
&= -kf''\frac{dk}{db} \\
&= -k\frac{dr}{db}
\end{aligned}$$

$$\begin{aligned}
f'(k) &= r \\
f''\frac{dk}{db} &= \frac{dr}{db}
\end{aligned}$$

The welfare effect of debt

$$s = k^s + (1+n)b = (1+n)(k+b)$$

$$\begin{aligned}\frac{1}{u_1}\frac{du^*}{db} &= \frac{dw}{db} - (r-n) \\ &\quad + \left[\frac{u_2}{u_1}s - b\right]\frac{dr}{db} \\ &= (n-r)\left(1 + \frac{b+k}{1+r}\frac{dr}{db}\right) \\ &< 0\end{aligned}$$