Exercise 2.1. *Howard's policy iteration algorithm*

Consider the Brock-Mirman problem: to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t,$$

subject to $c_t + k_{t+1} \leq A k_t^{\alpha} \theta_t$, k_0 given, $A > 0$, $1 > \alpha > 0$, where $\{\theta_t\}$ is an i.i.d. sequence with $\ln \theta_t$ distributed according to a normal distribution with mean zero and variance σ^2.
Consider the following algorithm. Guess at a policy of the form $k_{t+1} = h_0(A k_t^{\alpha} \theta_t)$ for any constant $h_0 \in (0,1)$. Then form

$$J_0(k_0, \theta_0) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(A k_t^{\alpha} \theta_t - h_0 A k_t^{\alpha} \theta_t).$$

Next choose a new policy h_1 by maximizing

$$\ln(A k^{\alpha} \theta - k') + \beta E J_0(k', \theta'),$$

where $k' = h_1 A k^{\alpha} \theta$. Then form

$$J_1(k_0, \theta_0) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(A k_t^{\alpha} \theta_t - h_1 A k_t^{\alpha} \theta_t).$$

Continue iterating on this scheme until successive h_j have converged.
Show that, for the present example, this algorithm converges to the optimal policy function in one step.

Solution

Under the policy $k_{t+1} = h_0 A k_t^{\alpha} \theta_t$, we get:

$$k_1 = h_0 A k_0^{\alpha} \theta_0 \text{ and } \ln k_1 = \ln A h_0 + \ln \theta_0 + \alpha \ln k_0.$$

Similarly, derive $\ln k_2, \ln k_3$...which yields the following recursive equation for $\ln k_t$:

$$\ln k_t = \ln (A h_0) \frac{1-\alpha^t}{1-\alpha} + \ln \theta_t + \alpha \ln \theta_{t-1} + \cdots + \alpha^{t-1} \ln \theta_0 + \alpha^t \ln k_0.$$

Plug this recursive formula for $\ln k_t$ into the objective function $E \sum_{t=0}^{\infty} \beta^t \ln (A k_t^{\alpha} \theta_t - h_0 A k_t^{\alpha} \theta_t)$ to derive $J_0(k_0, \theta_0)$:

$$\begin{aligned} J_0(k_0, \theta_0) &= \ln(1-h_0)A + \ln \theta_0 + \alpha \ln k_0 + \beta \left[\ln(1-h_0)A + E \ln \theta_1 + \alpha \ln k_1\right] \\ &\quad + \ldots \beta^t \left[\ln(1-h_0)A + E \ln \theta_t + \alpha \ln k_t\right] + \ldots \\ &= H_0 + H_1 \ln \theta_0 + \frac{\alpha}{1-\alpha\beta} \ln k_0, \end{aligned}$$

where H_0 and $H_1$are constants. Next, choose a policy h_1 to maximize

$$\begin{aligned} & \ln(Ak^\alpha\theta - k') + \beta E J_0(k_1, \theta_1) \\ = \; & \ln(Ak^\alpha\theta - k') + \beta E \left[H_0 + H_1 \ln\theta' + \frac{\alpha}{1-\alpha\beta}\ln k' \right]. \end{aligned}$$

The first-order condition for this problem is:

$$-\frac{1}{Ak^\alpha\theta - k'} + \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{k'} = 0,$$

which yields: $h_1 = \alpha\beta$. Now, plug the new policy function $k' = h_1 Ak^\alpha\theta$ into $E\sum_{t=0}^{\infty}\beta^t \ln\left(Ak_t^\alpha\theta_t - h_1 Ak_t^\alpha\theta_t\right)$ to derive $J_1(k_0, \theta_0)$. Firts, note that:

$$\ln k_t = \ln\left(Ah_1\right)\frac{1-\alpha^t}{1-\alpha} + \ln\theta_t + \alpha\ln\theta_{t-1} + \cdots + \alpha^{t-1}\ln\theta_0 + \alpha^t \ln k_0 \text{ for t} \geq 1.$$

Using this recursive formula, calculate $J_1(k_0, \theta_0)$:

$$J_1(k_0, \theta_0) = K_0 + K_1\ln\theta_0 + \frac{\alpha}{1-\alpha\beta}\ln k_0,$$

where K_0 and $K_1$are constants. Next, choose a policy h_2 to maximize

$$\begin{aligned} & \ln(Ak^\alpha\theta - k') + \beta E J_1(k_1, \theta_1) \\ = \; & \ln(Ak^\alpha\theta - k') + \beta E \left[K_0 + K_1 \ln\theta' + \frac{\alpha}{1-\alpha\beta}\ln k' \right]. \end{aligned}$$

The first-order condition for this problem is:

$$-\frac{1}{Ak^\alpha\theta - k'} + \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{k'} = 0,$$

which yields: $h_2 = \alpha\beta$. That's exactly what he had obtained for h_1! We have verified that our improvement algoritm has in fact converged after just one iteration.