Supplementary Appendix: Numerical Algorithm and Details of Derivations

1 Sketch of the Numerical Algorithm

The numerical algorithm used for calculating an equilibrium can be summarized as follows:
A. The normal distributions of $\ln \gamma$ and $\ln n_i$ are truncated at 6 standard deviations of their mean, and their densities are approximated using piecewise-linear functions with 120 nodes. Similarly, the joint density of (γ, n_i) and the equilibrium functions $p(\gamma, n_i)$, $q^b(\gamma, n_i)$, $q^s(\gamma, n_i)$, and $E(\gamma^{-1}|p_i)$ are approximated using piecewise-log-linear functions with 120 by 120 nodes.
B. Let $\Omega_i \equiv n_i \min \left\{ (\gamma\beta^{-1})^{\frac{1}{\eta}} p_i^{1-\frac{1}{\eta}}, z^b\gamma \right\}$. Substituting equations (13) and (19) in the paper into (10) implies that there is a one-to-one function mapping p_i onto a Ω_i, so observing p_i is equivalent to observing Ω_i. Given a value z^b and a function $p_i = p(\gamma, n_i)$, the density function of Ω_i can be directly calculated using its definition. Hence, using Bayes rule, one can calculate the density of γ conditional on Ω_i, and so $E(\gamma^{-1}|p_i) = E(\gamma^{-1}|\Omega_i)$.
C. Given the function $E(\gamma^{-1}|p_i)$, conditions (9), (10), (13), and (19) determine $p(\gamma, n_i)$, $q^b(\gamma, n_i)$ $q^s(\gamma, n_i)$, and z^b.
D. Our algorithm to calculate an equilibrium starts with a guess for z^b and $p(\gamma, n_i)$ (e.g. their values in a deterministic equilibrium). Using this guess, $E(\gamma^{-1}|p_i)$ is calculated using the method outlined in B. Then, using $E(\gamma^{-1}|p_i)$, new values for z^b and $p(\gamma, n_i)$ are calculated using the method outlined in C. Using these new values, our guess is revised with a weighted average of the original guess and the new values. This process is then continued until convergence is achieved.
E. Aggregate output is constructed integrating over n_i and adding the output in the centralized market.

The MATLAB files with the programs used are available from the authors upon request.

2 Euler Condition of the Demand for Money

To characterize the optimal demand for money, it is convenient to use the day budget (1) and the night Bellman equation (3) to eliminate y^b and V_t^b from the day optimization program (2). With this transformation, the first order condition that implicitly characterizes z_t^b is:

$$\int \left(\beta\gamma_t^{-1} + \pi\varrho_{it}\right) d\Phi(p_{it}, \gamma_t) = 1, \tag{A1}$$

where ϱ_{it} is the Lagrange multiplier associated with (4). The Lagrange multiplier ϱ_{it} can be solved from the first order condition for the optimal choice of q_t^b in (8):

$$\varrho_{it} = \frac{u'\left[\tilde{q}^b\left(z_t^b, \gamma_t, p_{it}, \Delta_t\right)\right]}{p_{it}} - \frac{\beta}{\gamma_t}. \tag{A2}$$

Using (A2), the condition (A1) for optimal demand for money can be conveniently transformed to:

$$\int \left\{\pi \frac{u'\left[\tilde{q}^b\left(z_t^b, \gamma_t, p_{it}, \Delta_t\right)\right]}{p_{it}} + (1-\pi)\frac{\beta}{\gamma_t}\right\} d\Phi(p_{it}, \gamma_t) = 1. \tag{A3}$$

That is, buyers equate the expected marginal benefit of z_t^b at night with the marginal cost of its acquisition during the day. If a buyer has a trading opportunity at night, the marginal value of money is the marginal utility of the goods he can purchase with an extra dollar, regardless of being cash constrained or not. If the buyer does not have a trading opportunity, the marginal value of money is the discounted marginal utility of the day good he can purchase tomorrow. Consequently, the expected marginal benefit of z_t^b in utils is the integral in the left-hand side of (A3). To acquire an extra real unit of money, the buyer must supply an extra unit of y_t^b, which costs one util, as stated in the right-hand side of (A3).

3 Response of Output to Expected Inflation

Since the day market is not affected by monetary shocks and all night markets face the same γ, both aggregate output and aggregate inflation are correlated with γ. Therefore, as emphasized by Lucas (1973), this model generates a short-run upward sloping Phillips curve. However, if monetary authorities were to increase the average rate of inflation by increasing μ_γ, aggregate output would actually fall. This negative effect of expected inflation on output in the special case of logarithmic preferences can be calculated analytically as follows.

Given the money demand in (12) and that γ is log-normally distributed, the expected output from (18) in the night market is:

$$Eq_i^s = AE(n_i^\lambda)\left(\frac{\pi}{\exp\left(\mu_\gamma - \frac{\sigma_\gamma^2}{2}\right) - (1-\pi)\,\beta}\right)^{\frac{1}{1+\alpha}} \exp\left(\lambda\mu_\gamma + \lambda^2\frac{\sigma_\gamma^2}{2}\right). \tag{A4}$$

Recall that $A = \left[\beta\exp(-\theta\mu_\gamma)^{\frac{1}{1+\alpha}}\right]$, $\lambda = \theta/(1+\alpha)$, and $\theta = \sigma_n^2/\left(\sigma_n^2 + \sigma_\gamma^2\right)$. Therefore,

$$\frac{d\ln Eq_i^s}{d\mu_\gamma} = -\frac{1}{1+\alpha}\frac{\exp\left(\mu_\gamma - \frac{\sigma_\gamma^2}{2}\right)}{\exp\left(\mu_\gamma - \frac{\sigma_\gamma^2}{2}\right) - (1-\pi)\,\beta} < 0. \tag{A5}$$

This expression is negative because $\exp\left(\mu_\gamma - \sigma_\gamma^2/2\right) - (1-\pi)\,\beta > 0$. Intuitively, with higher expected inflation, buyers carry less money balances, so output at night falls. This effect of expected inflation on output is not present in Lucas's contributions because with proportional transfers perfectly anticipated inflation has no effect on the demand for money.

4 Effect of Increasing the Variability of the Money Supply

Assuming logarithmic preferences, this part of the appendix shows the welfare effect of changing σ_γ^2 and adjusting μ_γ to keep the opportunity cost of holding money ($E\gamma^{-1}$) unchanged. Using $A = [\beta \exp(-\theta\mu_\gamma)]^{\frac{1}{1+\alpha}}$, $\lambda = \theta/(1+\alpha)$, and $\theta = \sigma_n^2/\left(\sigma_n^2 + \sigma_\gamma^2\right)$, we have that the aggregate utility buyers get from consumption at night is:

$$En_i \ln q^b = \frac{1}{1+\alpha} \ln z^b + \frac{\ln \beta}{1+\alpha} - \left(1 - \frac{1}{1+\alpha}\frac{\sigma_n^2}{\sigma_\gamma^2 + \sigma_n^2}\right) E\left(n_i \ln n_i\right) \tag{A6}$$

Likewise, the aggregate expected cost of production at night is:

$$E\frac{\left(q^s\right)^{1+\alpha}}{1+\alpha} = \frac{1}{1+\alpha} z^b \beta. \tag{A7}$$

Therefore, the aggregate expected surplus from trades at night is:

$$S = \frac{1}{1+\alpha} \ln z^b + \frac{\ln \beta}{1+\alpha} - \frac{1}{1+\alpha} z^b \beta - \left(1 - \frac{1}{1+\alpha}\frac{\sigma_n^2}{\sigma_\gamma^2 + \sigma_n^2}\right) E\left(n_i \ln n_i\right). \tag{A8}$$

Since economic activity during the day is not affected by monetary policy, and the demand for money in (13) implies that z^b is not affected by changes in σ_γ^2 if $E\gamma^{-1}$ is not changed, we have that the welfare effect of changing is σ_γ^2 and keeping $E\gamma^{-1}$ unchanged is:

$$\begin{aligned} \frac{dS}{d\sigma_\gamma^2} &= \frac{E\left(n_i \ln n_i\right)}{1+\alpha}\left[\frac{d}{d\sigma_\gamma^2}\left(\frac{\sigma_n^2}{\sigma_\gamma^2 + \sigma_n^2}\right)\right] \\ &= -\frac{E\left(n_i \ln n_i\right)}{1+\alpha}\frac{\sigma_n^2}{\left(\sigma_\gamma^2 + \sigma_n^2\right)^2}. \end{aligned} \tag{A9}$$

Given that $E\left(n_i \ln n_i\right) = \mu_n + cov\left(n_i, \ln n_i\right) > 0$, (A9) implies that an increase in σ_γ^2 is always detrimental for welfare with logarithmic preferences. As we show with the numerical analysis in Section 6, this result does not hold when preferences are not logarithmic because then buyers hold precautionary balances and an increase in σ_γ^2 may increase the demand for money which tends to correct the effect of a positive opportunity cost of holding money.