Measure theory and Perron-Frobenius operators for continued fractions

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1 The continued fraction transformation

For $\xi \in \mathbb{R}$ let $[x]$ be the greatest integer $\leq \xi$, let $R(\xi) = \xi - [\xi]$, and let $\|\xi\| = \min(R(\xi), 1-R(\xi))$, the distance from ξ to a nearest integer. Let $I = [0,1]$ and define the **continued fraction transformation** $\tau : I \to I$ by

$$\tau(x) = \begin{cases} x^{-1} - [x^{-1}] & x \neq 0 \\ 0 & x = 0. \end{cases}$$

It is immediate that for $x \in I$, $x \in I \setminus \mathbb{Q}$ if and only if $\tau(x) \in I \setminus \mathbb{Q}$. For $x \in \mathbb{R}$, define $a_0(x) = [x]$, and for $n \geq 1$ define $a_n(x) \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ by

$$a_n(x) = \left[\frac{1}{\tau^{n-1}(x - a_0(x))} \right].$$

For example, let $x = \frac{13}{71}$.

$$\tau(x) = \frac{71}{13} - \left[\frac{71}{13}\right] = \frac{71}{13} - 5 = \frac{6}{13}.$$

$$\tau^2(x) = \frac{13}{6} - \left[\frac{13}{6}\right] = \frac{13}{6} - 2 = \frac{1}{6}.$$

$$\tau^3(x) = \frac{6}{1} - \left[\frac{6}{1}\right] = 0.$$

Then $\tau^n(x) = 0$ for $n \geq 3$. Thus, with $x = \frac{13}{71}$,

$$a_0(x) = 0, \quad a_1(x) = \left[\frac{71}{13}\right] = 5.$$

$$a_2(x) = \left[\frac{1}{\tau(x)}\right] = \left[\frac{13}{6}\right] = 2, \quad a_3(x) = \left[\frac{1}{\tau^2(x)}\right] = \left[\frac{6}{1}\right] = 6.$$

$$a_4(x) = \left[\frac{1}{\tau^3(x)}\right] = \infty, \qquad a_5(x) = \infty, \qquad \ldots.$$

2 Convergents

For $x \in \Omega = I \setminus \mathbb{Q}$ write $a_n = a_n(x)$, and define

$$q_{-1} = 0, \quad p_{-1} = 1, \quad q_0 = 1, \quad p_0 = 0,$$

and for $n \geq 1$,

$$q_n = a_n q_{n-1} + q_{n-2}, \qquad p_n = a_n p_{n-1} + p_{n-2}.$$

Thus

$$q_1 = a_1 q_0 + q_{-1} = a_1, \qquad p_1 = a_1 p_0 + p_{-1} = 1.$$

One proves

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}, \qquad n \geq 0.$$

Also,[1]

$$x = \frac{p_n + \tau^n(x) p_{n-1}}{q_n + \tau^n(x) q_{n-1}}, \qquad x \in \Omega, \quad n \geq 0.$$

From this,

$$x - \frac{p_n}{q_n} = \frac{(-1)^n \tau^n(x)}{q_n(q_n + \tau^n(x) q_{n-1})}.$$

Now,

$$a_{n+1} + \tau^{n+1}(x) = \left[\frac{1}{\tau^n(x)}\right] + \frac{1}{\tau^n(x)} - \left[\frac{1}{\tau^n(x)}\right] = \frac{1}{\tau^n(x)},$$

and using this,

$$\begin{aligned} \frac{\tau^n(x)}{q_n(q_n + \tau^n(x) q_{n-1})} &= \frac{1}{q_n(q_n \cdot (a_{n+1} + \tau^{n+1}(x)) + q_{n-1})} \\ &= \frac{1}{q_n(q_{n+1} + \tau^{n+1}(x) q_n)}. \end{aligned}$$

Thus

$$\frac{1}{q_n(q_n + q_{n-1})} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

For $n \geq 1$ let

$$r_n(x) = \frac{1}{\tau^{n-1}(x)} = a_n + \tau^n(x)$$

and

$$s_n = \frac{q_{n-1}}{q_n}, \qquad y_n = \frac{1}{s_n}$$

[1] Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 9, Proposition 1.1.1.

and

$$\begin{aligned} u_n &= q_{n-1}^{-2}\left|x - \frac{p_{n-1}}{q_{n-1}}\right|^{-1} \\ &= \frac{1}{q_{n-1}^2} \cdot \frac{q_{n-1}(q_{n-1} + \tau^{n-1}(x)q_{n-2})}{\tau^{n-1}(x)} \\ &= \frac{q_{n-1} + \tau^{n-1}(x)q_{n-2}}{\tau^{n-1}(x)q_{n-1}} \\ &= \frac{q_{n-1} \cdot (a_n + \tau^n(x)) + q_{n-2}}{q_{n-1}} \\ &= a_n + \tau^n(x) + \frac{q_{n-2}}{q_{n-1}}. \end{aligned}$$

Let $s_0 = 0$. It is worth noting that

$$y_1 \cdots y_n = \frac{q_1}{q_0} \cdots \frac{q_n}{q_{n-1}} = \frac{q_n}{q_0} = q_n.$$

$$\frac{1}{s_n} = \frac{q_n}{q_{n-1}} = a_n + \frac{q_{n-2}}{q_{n-1}} = a_n + s_{n-1}.$$

$$u_n = a_n + \tau^n(x) + \frac{q_{n-2}}{q_{n-1}} = r_n + s_{n-1}.$$

3 Measure theory

Suppose that $(X, \mathscr{A})$ is a measurable space and μ, ν are probability measures on $\mathscr{A}$. Let $\mathscr{D} = \{A \in \mathscr{A} : \mu(A) = \nu(A)\}$. First, $X \in \mathscr{D}$. Second, if $A, B \in \mathscr{D}$ and $A \subset B$ then

$$\mu(B \setminus A) = \mu(B) - \mu(A) = \nu(B) - \nu(A) = \nu(B \setminus A),$$

so $B \setminus A \in \mathscr{D}$. Third, suppose that $A_n \in \mathscr{D}$, $n \geq 1$, and $A_n \uparrow A$. Because $\mathscr{A}$ is a σ-algebra, $A \in \mathscr{A}$, and then, setting $A_0 = \emptyset$,

$$\mu(A) = \mu\left(\bigcup_{n \geq 1}(A_n \setminus A_{n-1})\right) = \sum_{n \geq 1}(\mu(A_n) - \mu(A_{n-1})),$$

whence $\mu(A) = \nu(A)$. Therefore $\mathscr{D}$ is a Dynkin system. **Dynkin's theorem** says that if $\mathscr{D}$ is a Dynkin system and $\mathscr{C} \subset \mathscr{D}$ where $\mathscr{C}$ is a π-system (nonempty and closed under finite intersections), then $\sigma(\mathscr{C}) \subset \mathscr{D}$.[2]

Suppose now that $\sigma(\mathscr{C}) = \mathscr{A}$, that $\mathscr{C}$ is closed under finite intersections, and that $\mu(A) = \nu(A)$ for all $A \in \mathscr{C}$. Then $\mathscr{C} \subset \mathscr{D}$, so by Dynkin's theorem,

[2] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 136, Lemma 4.11.

$\mathscr{A} = \sigma(\mathscr{C}) \subset \mathscr{D}$, hence $\mathscr{D} = \mathscr{A}$. That is, for any $A \in \mathscr{A}$, $\mu(A) = \nu(A)$, meaning $\mu = \nu$.

We shall apply the above with $(I, \mathscr{B}_I)$, $I = [0, 1]$. For

$$\mathscr{C} = \{(0, u] : 0 < u \leq 1\},$$

it is a fact that $\sigma(\mathscr{C}) = \mathscr{B}_I$. Therefore if μ and ν are probability measures on $\mathscr{B}_I$ such that $\mu((0, u]) = \nu((0, u])$ for every $0 < u \leq 1$, then $\mu = \nu$.

Let λ be Lebesgue measure on $I = [0, 1]$. Define

$$d\gamma(x) = \frac{1}{(1+x)\log 2} d\lambda(x),$$

called the **Gauss measure**. If μ is a Borel probability measure on I, for measurable $T : I \to I$ and for $A \in \mathscr{B}_I$ let

$$T_*\mu(A) = \mu(T^{-1}(A)).$$

$T_*\mu$, called the **pushforward of μ by T**, is itself a Borel probability measure on I. We prove that γ is an invariant measure for τ.[3]

Theorem 1. *$\tau_*\gamma = \gamma$.*

Proof. Let $0 < u \leq 1$. For $x \in I$, $0 < \tau(x) \leq u$ if and only if $0 < \frac{1}{x} - \left[\frac{1}{x}\right] \leq u$ if and only if $\left[\frac{1}{x}\right] < \frac{1}{x} \leq u + \left[\frac{1}{x}\right]$ if and only if $\frac{1}{u+\left[\frac{1}{x}\right]} \leq x < \frac{1}{\left[\frac{1}{x}\right]}$. Then, as $0 \notin \tau^{-1}((0, u])$,

$$\tau^{-1}((0, u]) = \bigcup_{i \geq 1} \left[\frac{1}{u+i}, \frac{1}{i}\right).$$

We calculate

$$\begin{aligned}
\gamma(\tau^{-1}((0, u])) &= \sum_{i \geq 1} \gamma\left(\left[\frac{1}{u+i}, \frac{1}{i}\right)\right) \\
&= \sum_{i \geq 1} \int_{\left[\frac{1}{u+i}, \frac{1}{i}\right)} \frac{1}{(1+x)\log 2} d\lambda(x) \\
&= \frac{1}{\log 2} \sum_{i \geq 1} \left(\log\left(1 + \frac{1}{i}\right) - \log\left(1 + \frac{1}{u+i}\right)\right).
\end{aligned}$$

Using

$$\frac{1 + \frac{1}{i}}{1 + \frac{1}{u+i}} = \frac{1 + \frac{u}{i}}{1 + \frac{u}{i+1}},$$

[3] Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 17, Theorem 1.2.1; Manfred Einsiedler and Thomas Ward, *Ergodic Theory with a view towards Number Theory*, p. 77, Lemma 3.5.

this is

$$\begin{aligned}
\gamma(\tau^{-1}((0,u])) &= \frac{1}{\log 2}\sum_{i\geq 1}\left(\log\left(1+\frac{u}{i}\right)-\log\left(1+\frac{u}{i+1}\right)\right)\\
&= \frac{1}{\log 2}\sum_{i\geq 1}\int_{\frac{u}{i+1}}^{\frac{u}{i}}\frac{1}{1+x}d\lambda(x)\\
&= \gamma((0,u]).
\end{aligned}$$

Because $\gamma(\tau^{-1}((0,u])) = \gamma((0,u])$ for every $0 < u \leq 1$, it follows that $\tau_*\gamma = \gamma$. $\square$

We remark that for a set X, X^0 is a singleton. For $i \in \mathbb{Z}_{\geq 1}^0$ let $I_0(i) = \Omega$. For $n \geq 1$ and $i \in \mathbb{Z}_{\geq 1}^n$, let

$$I_n(i) = \{\omega \in \Omega : a_k(x) = i_k, 1 \leq k \leq n\}.$$

For $n \geq 1$ and for $i \in \mathbb{Z}_{\geq 1}^n$, define

$$[i_1,\dots,i_n] = \cfrac{1}{i_1 + \cfrac{1}{\cdots + \cfrac{1}{i_{n-1}+\cfrac{1}{i_n}}}}.$$

For $x \in I_n(i)$,

$$\frac{p_n(x)}{q_n(x)} = [i_1,\dots,i_n], \qquad \frac{p_{n-1}(x)}{q_{n-1}(x)} = [i_1,\dots,i_{n-1}].$$

The following is an expression for the sets $I_n(i)$.[4]

Theorem 2. *Let $n \geq 1$, $i \in \mathbb{Z}_{\geq 1}^n$, and define*

$$u_n(i) = \begin{cases} \frac{p_n+p_{n-1}}{q_n+q_{n-1}} & n \text{ odd}\\ \frac{p_n}{q_n} & n \text{ even}\end{cases}$$

and

$$v_n(i) = \begin{cases} \frac{p_n}{q_n} & n \text{ odd}\\ \frac{p_n+p_{n-1}}{q_n+q_{n-1}} & n \text{ even.}\end{cases}$$

Then

$$I_n(i) = \Omega \cap (u_n(i), v_n(i)).$$

[4]Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 18, Theorem 1.2.2.

From the above, if n is odd and $i \in \mathbb{Z}_{\geq 1}$ then

$$\begin{aligned}
\lambda(I_n(i)) &= v_n(i) - u_n(i) \\
&= \frac{p_n}{q_n} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \\
&= \frac{p_n q_{n-1} - p_{n-1} q_n}{q_n(q_n + q_{n-1})} \\
&= \frac{(-1)^{n+1}}{q_n(q_n + q_{n-1})} \\
&= \frac{1}{q_n(q_n + q_{n-1})},
\end{aligned}$$

and if n is even then likewise

$$\lambda(I_n(i)) = \frac{1}{q_n(q_n + q_{n-1})}.$$

Kraaikamp and Iosifescu attribute the following to Torsten Brodén, in a 1900 paper.[5]

Theorem 3. *For $n \geq 1$, $i \in \mathbb{N}^n$, $x \in I$,*

$$\lambda(\tau^n < x | i) = \frac{x(s_n + 1)}{s_n x + 1}.$$

Proof. We have

$$\lambda(\tau^n < x | i) = \frac{\lambda((\tau^n < x) \cap I_n(i))}{\lambda(I_n(i))}.$$

Using

$$\omega = \frac{p_n + \tau^n(\omega) p_{n-1}}{q_n + \tau^n(\omega) q_{n-1}}, \qquad \omega \in \Omega, \quad n \geq 0,$$

if n is odd then

$$\begin{aligned}
(\tau^n < x) \cap I_n(i) &= \left\{ \omega \in \Omega : \frac{p_n + p_{n-1}}{q_n + q_{n-1}} < \omega < \frac{p_n}{q_n}, \tau^n(\omega) < x \right\} \\
&= \left\{ \omega \in \Omega : \frac{p_n + x p_{n-1}}{q_n + x q_{n-1}} < \omega < \frac{p_n}{q_n} \right\}
\end{aligned}$$

and if n is even then

$$(\tau^n < x) \cap I_n(i) = \left\{ \omega \in \Omega : \frac{p_n}{q_n} < \omega < \frac{p_n + x p_{n-1}}{q_n + x q_{n-1}} \right\}.$$

[5]Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 21, Corollary 1.2.6.

Therefore if n is odd,

$$\begin{aligned}\lambda((\tau^n < x) \cap I_n(i)) &= \frac{p_n}{q_n} - \frac{p_n + xp_{n-1}}{q_n + xq_{n-1}} \\ &= \frac{xp_nq_{n-1} - xp_{n-1}q_n}{q_n(q_n + xq_{n-1})} \\ &= \frac{x}{q_n(q_n + xq_{n-1})}\end{aligned}$$

and likewise if n is even then

$$\lambda((\tau^n < x) \cap I_n(i)) = \frac{x}{q_n(q_n + xq_{n-1})}.$$

Therefore for $n \geq 1$,

$$\begin{aligned}\lambda(\tau^n < x|i) &= \frac{x}{q_n(q_n + xq_{n-1})} \cdot q_n(q_n + q_{n-1}) \\ &= \frac{x(q_n + q_{n-1})}{q_n + xq_{n-1}}.\end{aligned}$$

Using $s_n + 1 = \frac{q_n+q_{n-1}}{q_n}$ and $s_nx + 1 = \frac{xq_{n-1}+q_n}{q_n}$,

$$\begin{aligned}\lambda(\tau^n < x|i) &= \frac{xq_n(s_n + 1)}{q_n(s_nx + 1)} \\ &= \frac{x(s_n + 1)}{s_nx + 1}.\end{aligned}$$

□

For $j \geq 1$ and $s \in I$ define

$$P_j(s) = \frac{s + 1}{(s + j)(s + j + 1)}.$$

We now apply Theorem 3 to prove the following.[6]

Theorem 4. *For $j \geq 1$,*

$$\lambda(a_1 = j) = \frac{1}{j(j + 1)}.$$

For $n \geq 1$ and $i \in \mathbb{N}^n$,

$$\lambda(a_{n+1} = j|i) = P_j(s_n).$$

[6]Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 22, Proposition 1.2.7.

Proof. By Theorem 2,

$$\{\omega \in \Omega : a_1(\omega) = j\} = I_1(j) = \Omega \cap (u_1(j), v_1(j)).$$

In this case, $q_1 = j$, so $u_1(j) = \frac{p_1+p_0}{q_1+q_0} = \frac{1+0}{j+1} = \frac{1}{j+1}$ and $v_1(j) = \frac{p_1}{q_1} = \frac{1}{j}$, so

$$\{\omega \in \Omega : a_1(\omega) = j\} = \Omega \cap \left(\frac{1}{j+1}, \frac{1}{j}\right).$$

Now,

$$a_{n+1}(\omega) = \left[\frac{1}{\tau^n(\omega)}\right] = a_1(\tau^n(\omega)).$$

Thus

$$\{\omega \in \Omega : a_{n+1}(\omega) = j\} = \left\{\omega \in \Omega : \tau^n(\omega) \in \left(\frac{1}{j+1}, \frac{1}{j}\right)\right\}.$$

Then using Theorem 3,

$$\begin{aligned}
\lambda(a_{n+1} = j|i) &= \lambda\left(\tau^n < \frac{1}{j}|i\right) - \lambda\left(\tau^n < \frac{1}{j+1}|i\right) \\
&= \frac{\frac{1}{j}(s_n+1)}{s_n\frac{1}{j}+1} - \frac{\frac{1}{j+1}(s_n+1)}{s_n\frac{1}{j+1}+1} \\
&= \frac{s_n+1}{(s_n+1)(s_n+j+1)}.
\end{aligned}$$

□

4 Perron-Frobenius operators

For a probability measure μ on $\mathscr{B}_I$ and for $f \in L^1(\mu)$ let $d\mu_f = f d\mu$. If $\tau_*\mu$ is absolutely continuous with respect to μ, check that $\tau_*\mu_f$ is itself absolutely continuous with respect to μ. Then applying the Radon-Nikodym theorem, let

$$P_\mu f = \frac{d(\tau_*\mu_f)}{d\mu}.$$

For $g \in L^\infty(\mu)$,

$$\int_I g \cdot P_\mu f d\mu = \int_I g d(\tau_*\mu_f) = \int_I g \circ \tau d\mu_f = \int_I (g \circ \tau) \cdot f d\mu.$$

In particular, for $g = 1_A$, $A \in \mathscr{B}_I$,

$$\int_I 1_A \cdot P_\mu f d\mu = \int_I 1_{\tau^{-1}(A)} \cdot f d\mu.$$

For $g \in L^\infty(\mu)$,

$$\int_I g \cdot P_\gamma 1 d\gamma = \int_I g \circ \tau d\gamma = \int_I g d(\tau_*\gamma),$$

hence $P_\gamma 1 = 1$ if and only if $\tau_* \gamma$.

We shall be especially interested in

$$U = P_\gamma,$$

where γ is the Gauss measure on I. We establish almost everywhere an expression for $Uf(x)$.[7]

Theorem 5. *For $f \in L^1(\gamma)$, for γ-almost all $x \in I$,*

$$Uf(x) = \sum_{i \geq 1} P_i(x) f\left(\frac{1}{x+i}\right).$$

Proof. Let $I_i = \left(\frac{1}{i+1}, \frac{1}{i}\right]$ and let τ_i be the restriction of $\tau : I \to I$ to I_i. For $u \in I_i$, $i \leq \frac{1}{u} < i+1$, hence $\tau_i(u) = \tau(u) = \frac{1}{u} - i$, i.e. $u = \frac{1}{\tau_i(u)+i}$, i.e. $\tau_i^{-1}(x) = \frac{1}{x+i}$.

For $A \in \mathscr{B}_I$, if $0 \notin A$ then

$$\tau^{-1}(A) = \tau^{-1}\left(\bigcup_{i \geq 1}(A \cap I_i)\right) = \bigcup_{i \geq 1} \tau^{-1}(A \cap I_i),$$

and the sets $\tau^{-1}(A \cap I_i)$ are pairwise disjoint, hence

$$\int_{\tau^{-1}(A)} f d\gamma = \sum_{i \geq 1} \int_{\tau^{-1}(A \cap I_i)} f d\gamma = \sum_{i \geq 1} \int_{\tau_i^{-1}(A)} f d\gamma.$$

Applying the change of variables formula, as $\frac{d}{dx}\tau_i^{-1}(x) = -(x+i)^{-2}$,

$$\begin{aligned}
\int_{\tau_i^{-1}(A)} f d\gamma &= \frac{1}{\log 2} \int_{\tau_i^{-1}(A)} \frac{f(u)}{u+1} d\lambda(u) \\
&= \frac{1}{\log 2} \int_A \frac{f \circ \tau_i^{-1}(x)}{\tau_i^{-1}(x)+1} \cdot (x+i)^{-2} d\lambda(x) \\
&= \frac{1}{\log 2} \int_A f\left(\frac{1}{x+i}\right) \cdot \frac{1}{(x+i+1)(x+i)} d\lambda(x) \\
&= \frac{1}{\log 2} \int_A f\left(\frac{1}{x+i}\right) \cdot P_i(x) \cdot \frac{1}{x+1} d\lambda(x) \\
&= \int_A f\left(\frac{1}{x+i}\right) \cdot P_i(x) d\gamma(x).
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{\tau^{-1}(A)} f d\gamma &= \sum_{i \geq 1} \int_A f\left(\frac{1}{x+i}\right) \cdot P_i(x) d\gamma(x) \\
&= \int_A \sum_{i \geq 1} f\left(\frac{1}{x+i}\right) \cdot P_i(x) d\gamma(x).
\end{aligned}$$

[7]Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 59, Proposition 2.1.2.

Then

$$\int_A P_\gamma f d\gamma = \int_A \sum_{i\geq 1} f\left(\frac{1}{x+i}\right) \cdot P_i(x) d\gamma(x).$$

Because this is true for any $A \in \mathscr{B}_I$ with $0 \notin A$, it follows that for γ-almost all $x \in I$,

$$P_\gamma f(x) = \sum_{i\geq 1} f\left(\frac{1}{x+i}\right) \cdot P_i(x).$$

□

The following gives an expression for $P_\mu f(x)$ under some hypotheses.[8]

Theorem 6. *Let μ be a probability measure on $\mathscr{B}_I$ that is absolutely continuous with respect to λ and suppose that $d\mu = hd\lambda$ with $h(x) > 0$ for μ-almost all $x \in I$. Let $f \in L^1(\mu)$ and define $g(x) = (x+1)h(x)f(x)$. For μ-almost all $x \in I$,*

$$P_\mu f(x) = \frac{1}{h(x)} \sum_{i\geq 1} \frac{h((x+i)^{-1})}{(x+i)^2} f\left(\frac{1}{x+i}\right) = \frac{Ug(x)}{(x+1)h(x)}.$$

For $n \geq 1$, for μ-almost all $x \in I$,

$$P_\mu^n f(x) = \frac{U^n g(x)}{(x+1)h(x)}.$$

We prove an expression for $\mu(\tau^{-n}(A))$.[9]

Theorem 7. *Let μ be a probability measure on $\mathscr{B}_I$ that is absolutely continuous with respect to λ. Let $h = \frac{d\mu}{d\lambda}$ and let $f(x) = (x+1)h(x)$. For $A \in \mathscr{B}_I$ and $n \geq 1$,*

$$\mu(\tau^{-n}(A)) = \int_A \frac{U^n f(x)}{x+1} d\lambda(x).$$

Proof. For $n = 0$,

$$\mu(A) = \int_A d\mu = \int_A h d\lambda = \int_A \frac{f(x)}{x+1} d\lambda(x) = \int_A \frac{U^0 f(x)}{x+1} d\lambda(x).$$

[8] Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 60, Proposition 2.1.3.

[9] Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 61, Proposition 2.1.5.

Suppose by hypothesis that the claim is true for some $n \geq 0$. Then

$$\begin{aligned}
\mu(\tau^{-n-1}(A)) &= \mu(\tau^{-n}(\tau^{-1}(A))) \\
&= \int_{\tau^{-1}(A)} \frac{U^n f(x)}{x+1} d\lambda(x) \\
&= \log 2 \cdot \int_{\tau^{-1}(A)} U^n f(x) d\gamma(x) \\
&= \log 2 \cdot \int_A U^{n+1} f(x) d\gamma(x) \\
&= \log 2 \cdot \int_A \frac{U^{n+1} f(x)}{x+1} d\lambda(x).
\end{aligned}$$

□

For $f(x) = \frac{1}{x+1}$ and $A \in \mathscr{B}_I$,

$$\begin{aligned}
\int_A P_\lambda f d\lambda &= \int_{\tau^{-1}(A)} \frac{1}{x+1} d\lambda(x) \\
&= \log 2 \cdot \int_{\tau^{-1}(A)} d\gamma \\
&= \log 2 \cdot \int_A d\gamma \\
&= \int_A f d\lambda.
\end{aligned}$$

Because this is true for all Borel sets A,

$$P_\lambda \frac{1}{x+1} = \frac{1}{x+1}.$$

For $f \in L^1(\lambda)$ and $x \in I$, let

$$\Pi_1 f(x) = \frac{1}{(x+1)\log 2} \int_I f d\lambda.$$

Define

$$T_0 = P_\lambda - \Pi_1.$$

For $n \geq 1$, $\Pi_1^n = \Pi_1$. For $f \in L^1(\lambda)$,

$$P_\lambda \Pi_1 f = \frac{1}{\log 2} \int_I f d\lambda \cdot P_\lambda \frac{1}{x+1} = \frac{1}{\log 2} \int_I f d\lambda \cdot \frac{1}{x+1} = \Pi_1 f(x)$$

and

$$\Pi_1 P_\lambda f = \frac{1}{(x+1)\log 2} \int_I P_\lambda f d\lambda = \frac{1}{(x+1)\log 2} \int_I f d\lambda = \Pi_1 f(x),$$

hence

$$P_\lambda \Pi_1 = \Pi_1 = \Pi_1 P_\lambda.$$

Moreover,

$$T_0 \Pi_1 = (P_\lambda - \Pi_1)\Pi_1 = P_\lambda \Pi_1 - \Pi_1^2 = 0$$

and

$$\Pi_1 T_0 = \Pi_1 (P_\lambda - \Pi_1) = \Pi_1 P_\lambda - \Pi_1^2 = 0.$$

Because $P_\lambda = \Pi_1 + T_0$, using $\Pi_1^2 = \Pi_1$, $T_0 \Pi_1 = 0$, and $\Pi_1 T_0 = 0$, we have

$$P_\lambda^n = \Pi_1 + T_0^n, \qquad n \geq 1.$$

Theorem 6 tells us that for $f \in L^1(\lambda)$, for λ-almost all $x \in I$,

$$P_\lambda f(x) = \sum_{i \geq 1} \frac{1}{(x+i)^2} f\left(\frac{1}{x+i}\right).$$

With $h(x) = x+1$ and $g = hf$, for $n \geq 1$, for λ-almost all $x \in I$,

$$P_\lambda^n f(x) = \frac{U^n g(x)}{x+1}.$$

Thus

$$\begin{aligned} U^n g &= h P_\lambda^n f \\ &= h \Pi_1 f + h T_0^n f \\ &= \frac{1}{\log 2} \int_I f d\lambda + h T_0^n f \\ &= \int_I g d\gamma + h T_0^n (g/h). \end{aligned}$$

Define $I_\gamma : L^1(\gamma) \to L^1(\gamma)$ by

$$I_\gamma f = 1 \cdot \int_I f d\gamma.$$

We have

$$I_\gamma U f = \int_I P_\gamma f d\gamma = \int_I f d\gamma = I_\gamma f,$$

meaning $I_\gamma U = I_\gamma$. Furthermore, because $\tau_* \gamma = \gamma$ we have $P_\gamma 1 = 1$, so

$$U I_\gamma f = \int_I f d\gamma \cdot U1 = \int_I f d\gamma \cdot 1 = I_\gamma f,$$

meaning $U I_\gamma = I_\gamma$.

Let $h(x) = x + 1$. $h, \frac{1}{h} \in L^\infty(\gamma)$. Now define $T : L^1(\gamma) \to L^1(\gamma)$ by

$$Tg = h \cdot T_0(g/h),$$

which makes sense because $\frac{1}{h} \in L^\infty(\gamma)$. Then

$$\begin{aligned} T^2 g &= T(h \cdot T_0(g/h)) \\ &= h \cdot T_0 \left(\frac{h \cdot T_0(g/h)}{h} \right) \\ &= h \cdot T_0^2(g/h). \end{aligned}$$

For $n \geq 1$,

$$T^n g = h \cdot T_0^n(g/h).$$

Recapitulating the above, for $n \geq 1$ and $g \in L^1(\gamma)$,

$$U^n g = I_\gamma g + h T_0^n(g/h) = I_\gamma g + T^n g,$$

meaning

$$U^n = I_\gamma + T^n, \qquad n \geq 1.$$

It is a fact that T^n converges to 0 in the strong operator topology on $\mathscr{L}(L^1(\gamma))$, the bounded linear operators $L^1(\gamma) \to L^1(\gamma)$, that is, for each $f \in L^1(\gamma)$, $T^n f \to 0$ in $L^1(\gamma)$, i.e. $\|T^n f\|_{L^1} \to 0$.[10] Then $U^n \to I_\gamma$ in the strong operator topology: for $f \in L^1(\gamma)$,

$$\int_I \left| U^n f(x) - \int_I f d\gamma \right| d\lambda \to 0.$$

Iosifescu and Kraaikamp state that has not been determined whether for γ-almost all $x \in I$, $U_n f(x) \to I_\gamma f$.

Let $B(I)$ be the set of bounded Borel measurable functions $f : I \to \mathbb{C}$ and write $\|f\|_\infty = \sup_{x \in I} |f(x)|$. For $f \in B(I)$, define for $x \in I$,

$$Uf(x) = \sum_{i \geq 1} P_i(x) f\left(\frac{1}{x+i} \right) = \sum_{i \geq 1} \frac{x+1}{(x+i)(x+i+1)} f\left(\frac{1}{x+i} \right).$$

$1 \in B(I)$, and for $x \in I$,

$$\sum_{1 \leq i \leq m} \frac{x+1}{(x+i)(x+i+1)} = \frac{m}{m+x+1},$$

hence

$$U1(x) = \sum_{i \geq 1} \frac{x+1}{(x+i)(x+i+1)} = 1.$$

For $f \in B(I)$ and $x \in I$,

$$|Uf(x)| \leq \|f\|_\infty \cdot U1(x),$$

[10] Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 63, Proposition 2.1.7.

hence

$$\|U\|_{B(I)\to B(I)} = 1.$$

Say that $f : I \to \mathbb{R}$ is increasing if $x \le y$ implies $f(x) \le f(y)$. An increasing function $f : I \to \mathbb{R}$ belongs to $B(I)$. We prove that if f is increasing then Uf is decreasing.[11]

Theorem 8. *If $f : I \to \mathbb{R}$ is increasing then Uf is decreasing.*

Proof. Take $x < y$ and let

$$S_1 = \sum_{i\ge 1} P_i(y)\left(f\left(\frac{1}{y+i}\right) - f\left(\frac{1}{x+i}\right)\right)$$

and

$$S_2 = \sum_{i\ge 1} (P_i(y) - P_i(x)) f\left(\frac{1}{x+i}\right).$$

Then

$$\begin{aligned} Uf(y) - Uf(x) &= \sum_{i\ge 1}\left(P_i(y) f\left(\frac{1}{y+i}\right) - P_i(x) f\left(\frac{1}{x+i}\right)\right) \\ &= S_1 + S_2. \end{aligned}$$

Because f is increasing, $S_1 \le 0$. Using $\sum_{i\ge 1} P_i(u) = 1$ for any $u \in I$,

$$\sum_{i\ge 1} (P_i(y) - P_i(x)) f\left(\frac{1}{x+1}\right) = 0,$$

and therefore

$$\begin{aligned} S_2 &= \sum_{i\ge 1}\left(f\left(\frac{1}{x+i}\right) - f\left(\frac{1}{x+1}\right)\right)(P_i(y) - P_i(x)) \\ &= \left(f\left(\frac{1}{x+2}\right) - f\left(\frac{1}{x+1}\right)\right)(P_2(y) - P_2(x)) \\ &\quad + \sum_{i\ge 3}\left(f\left(\frac{1}{x+i}\right) - f\left(\frac{1}{x+1}\right)\right)(P_i(y) - P_i(x)). \end{aligned}$$

For $i \ge 2$, using that f is increasing,

$$f\left(\frac{1}{x+i}\right) - f\left(\frac{1}{x+1}\right) \le f\left(\frac{1}{x+2}\right) - f\left(\frac{1}{x+1}\right) \le 0.$$

We calculate

$$P_i'(u) = -\frac{-i^2 + i + (u+1)^2}{(u+i)^2(u+i+1)^2}.$$

[11] Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 65, Proposition 2.1.11.

The roots of the above rational function are $u = -\sqrt{(i-1)i} - 1, \sqrt{(i-1)i} - 1$. Thus, $P_i'(u) = 0$ if and only if $u = \sqrt{(i-1)i} - 1$. But $\sqrt{(i-1)i} - 1 \in I$ if and only if $i^2 - i - 1 \geq 0$ and $i^2 - i - 4 \leq 0$. This is possible if and only if $i = 2$. And

$$P_i'(0) = \frac{i^2 - i - 1}{i^2(i+1)^2},$$

so $P_1'(u) \leq 0$ for all $u \in I$ and for $i \geq 3$, $P_i'(u) \geq 0$ for all $u \in I$. For $i = 2$, check that if $0 \leq u \leq \sqrt{2} - 1$ then $P_2'(u) \geq 0$ and if $\sqrt{2} - 1 \leq u \leq 1$ then $P_2'(u) \leq 0$. Then

$$\begin{aligned}
S_2 &\leq \left(f\left(\frac{1}{x+2}\right) - f\left(\frac{1}{x+1}\right) \right) (P_2(y) - P_2(x)) \\
&\quad + \sum_{i \geq 3} \left(f\left(\frac{1}{x+2}\right) - f\left(\frac{1}{x+1}\right) \right) (P_i(y) - P_i(x)) \\
&= \left(f\left(\frac{1}{x+2}\right) - f\left(\frac{1}{x+1}\right) \right) (P_2(y) - P_2(x)) \\
&\quad + \left(f\left(\frac{1}{x+2}\right) - f\left(\frac{1}{x+1}\right) \right) (-P_1(y) - P_2(y) - (-P_1(x) - P_2(x))) \\
&= \left(f\left(\frac{1}{x+2}\right) - f\left(\frac{1}{x+1}\right) \right) (P_1(x) - P_1(y)) \\
&\leq 0.
\end{aligned}$$

We have shown that $S_1 \leq 0$ and $S_2 \leq 0$, so

$$Uf(y) - Uf(x) = S_1 + S_2 \leq 0,$$

which means that $Uf : I \to \mathbb{R}$ is decreasing. □

For $J = [a, b] \subset I$, a **partition** of J is a sequence $P = (t_0, \ldots, t_n)$ such that $a = t_0 < \cdots < t_n = b$. For $f : I \to \mathbb{R}$ define[12]

$$V(f, P) = \sum_{1 \leq i \leq n} |f(t_i) - f(t_{i-1})|.$$

Define

$$V_J f = \sup\{V(f, P) : P \text{ is a partition of } J\}.$$

Let $v_f(x) = V_{[0,x]} f$, the **variation of** f. $v_f(1) = V_{[0,1]} f$. We say that f has **bounded variation** if $v_f(1) < \infty$, and denote by $BV(I)$ the set of functions $f : I \to \mathbb{R}$ with bounded variation. It is a fact that with the norm

$$\|f\|_{BV} = |f(0)| + V_I f,$$

$BV(I)$ is a Banach algebra.[13]

[12] `http://individual.utoronto.ca/jordanbell/notes/helly.pdf`

[13] `http://individual.utoronto.ca/jordanbell/notes/helly.pdf`, Theorem 11.

If f is increasing then $V_I f = f(1) - f(0)$. We will use the following to prove the theorem coming after it.[14]

Lemma 9. *If $f : I \to \mathbb{R}$ is increasing then*

$$V_I(Uf) \leq \frac{1}{2} V_I f.$$

Proof. Because Uf is decreasing,

$$V_I(Uf) = Uf(0) - Uf(1) = \sum_{i\geq 1} \left(P_i(0) f\left(\frac{1}{i}\right) - P_i(1) f\left(\frac{1}{1+i}\right)\right).$$

As $P_i(u) = \frac{u+1}{(u+i)(u+i+1)}$,

$$P_i(1) = \frac{2}{(i+1)(i+2)} = 2P_{i+1}(0),$$

hence

$$\begin{aligned} V_I(Uf) &= \sum_{i\geq 1} \left(P_i(0) f\left(\frac{1}{i}\right) - P_i(1) f\left(\frac{1}{1+i}\right)\right) \\ &= \sum_{i\geq 1} \left(P_i(0) f\left(\frac{1}{i}\right) - P_{i+1}(0) f\left(\frac{1}{1+i}\right)\right) \\ &- \sum_{i\geq 1} P_{i+1}(0) f\left(\frac{1}{1+i}\right) \\ &= P_1(0) f(1) - \sum_{i\geq 1} P_{i+1}(0) f\left(\frac{1}{1+i}\right) \\ &= \frac{1}{2} f(1) - \sum_{i\geq 1} P_{i+1}(0) f\left(\frac{1}{1+i}\right). \end{aligned}$$

Because $f\left(\frac{1}{1+i}\right) \geq f(0)$ we have $-f\left(\frac{1}{1+i}\right) \leq -f(0)$, hence

$$V_I(Uf) \leq \frac{1}{2} f(1) - f(0) \sum_{i\geq 1} P_{i+1}(0) = \frac{1}{2} f(1) - \frac{1}{2} f(0),$$

using $\sum_{i\geq 1} P_i(0) = 1$ and $P_1(0) = \frac{1}{2}$. As f is increasing this means

$$V_I(Uf) \leq \frac{1}{2}(f(1) - f(0)) = \frac{1}{2} V_I f.$$

□

[14] Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 66, Proposition 2.1.12.

Theorem 10. *If $f \in BV(I)$ then*

$$V_I(Uf) \le \frac{1}{2} V_I f.$$

Proof. Let

$$p_f(x) = \frac{v_f(x) + f(x) - f(0)}{2}, \qquad n_f(x) = \frac{v_f(x) - f(x) + f(0)}{2},$$

the **positive variation** of f and the **negative variation** of f. It is a fact that $0 \le p_f \le v_f$, $0 \le n_f \le v_f$, and p_f and n_f are increasing.[15] Using this,

$$\begin{aligned} V_I(Uf) &= V_I(Up_f + Un_f) \\ &\le \frac{1}{2} V_I p_f + \frac{1}{2} V_I n_f \\ &= \frac{1}{2}(p_f(1) - p_f(0)) + \frac{1}{2}(n_f(1) - n_f(0)) \\ &= \frac{1}{2}(v_f(1) - v_f(0)) \\ &= \frac{1}{2} V_I f. \end{aligned}$$

□

For $f : I \to \mathbb{C}$, let

$$s(f) = \sup_{x,y \in I, x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

We denote by $\mathrm{Lip}(I)$ the set of $f : I \to \mathbb{C}$ such that $s(f) < \infty$.[16]

Theorem 11. *For $f \in \mathrm{Lip}(I)$,*

$$s(Uf) \le (2\zeta(3) - \zeta(2)) s(f).$$

[15] `http://individual.utoronto.ca/jordanbell/notes/helly.pdf`, Theorem 10.

[16] Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 67, Proposition 2.1.14.

Proof. Suppose $x, y \in I$, $x > y$. We calculate

$$\begin{aligned}
&\frac{Uf(y) - Uf(x)}{y - x} \\
=&\frac{1}{y-x}\sum_{i\geq 1}\left(P_i(y)f\left(\frac{1}{y+i}\right) - P_i(y)f\left(\frac{1}{x+i}\right)\right) \\
&+ \frac{1}{y-x}\sum_{i\geq 1}\left(P_i(y)f\left(\frac{1}{x+i}\right) - P_i(x)f\left(\frac{1}{x+i}\right)\right) \\
=&\sum_{i\geq 1} P_i(y)\cdot\frac{f\left(\frac{1}{y+i}\right) - f\left(\frac{1}{x+i}\right)}{y-x} \\
&+ \sum_{i\geq 1}\frac{P_i(y) - P_i(x)}{y-x} f\left(\frac{1}{x+i}\right).
\end{aligned}$$

Calculating further,

$$\begin{aligned}
\frac{Uf(y) - Uf(x)}{y-x} = &-\sum_{i\geq 1} P_i(y)\cdot\frac{f\left(\frac{1}{y+i}\right) - f\left(\frac{1}{x+i}\right)}{\frac{1}{y+i} - \frac{1}{x+i}}\cdot\frac{1}{(x+i)(y+i)} \\
&+ \sum_{i\geq 1}\frac{P_i(y) - P_i(x)}{y-x} f\left(\frac{1}{x+i}\right).
\end{aligned}$$

Now,

$$P_i(u) = \frac{u+1}{(u+i)(u+i+1)} = \frac{i}{u+i+1} - \frac{i-1}{u+i},$$

whence

$$P_i(y) - P_i(x) = \frac{(x-y)i}{(x+i+1)(y+i+1)} + \frac{(y-x)(i-1)}{(x+i)(y+i)},$$

therefore

$$\begin{aligned}
&\sum_{i\geq 1}\frac{P_i(y) - P_i(x)}{y-x} f\left(\frac{1}{x+i}\right) \\
=&\sum_{i\geq 1}\left(\frac{i-1}{(x+i)(y+i)} - \frac{i}{(x+i+1)(y+i+1)}\right) f\left(\frac{1}{x+i}\right).
\end{aligned}$$

Summation by parts tells us

$$\sum_{i\geq 1} f_i(g_{i+1} - g_i) = -f_1 g_1 - \sum_{i\geq 1} g_{i+1}(f_{i+1} - f_i),$$

and here this yields, for $g_i = \frac{i-1}{(x+i)(y+i)}$ and $f_i = f\left(\frac{1}{x+i}\right)$,

$$\begin{aligned}
&\sum_{i\geq 1}\left(\frac{i-1}{(x+i)(y+i)} - \frac{i}{(x+i+1)(y+i+1)}\right) f\left(\frac{1}{x+i}\right)\\
&= \sum_{i\geq 1} g_{i+1}(f_{i+1} - f_i)\\
&= \sum_{i\geq 1} \frac{i}{(x+i+1)(y+i+1)}\left(f\left(\frac{1}{x+i+1}\right) - f\left(\frac{1}{x+i}\right)\right)\\
&= \sum_{i\geq 1} \frac{i}{(x+i+1)(y+i+1)} \cdot \frac{f\left(\frac{1}{x+i+1}\right) - f\left(\frac{1}{x+i}\right)}{\frac{1}{x+i+1} - \frac{1}{x+i}} \cdot \frac{-1}{(x+i)(x+i+1)}.
\end{aligned}$$

Recapitulating the above,

$$\begin{aligned}
&\frac{Uf(y) - Uf(x)}{y-x}\\
&= -\sum_{i\geq 1} P_i(y) \cdot \frac{f\left(\frac{1}{y+i}\right) - f\left(\frac{1}{x+i}\right)}{\frac{1}{y+i} - \frac{1}{x+i}} \cdot \frac{1}{(x+i)(y+i)}\\
&\quad - \sum_{i\geq 1} \frac{i}{(x+i)(x+i+1)^2(y+i+1)} \cdot \frac{f\left(\frac{1}{x+i+1}\right) - f\left(\frac{1}{x+i}\right)}{\frac{1}{x+i+1} - \frac{1}{x+i}}.
\end{aligned}$$

Then

$$\begin{aligned}
\left|\frac{Uf(y) - Uf(x)}{y-x}\right| &\leq s(f) \sum_{i\geq 1} P_i(y) \frac{1}{(x+i)(y+i)}\\
&\quad + s(f) \sum_{i\geq 1} \frac{i}{(x+i)(x+i+1)^2(y+i+1)}.
\end{aligned}$$

Then, using that $x > y$,

$$\left|\frac{Uf(y) - Uf(x)}{y-x}\right| \leq s(f) \sum_{i\geq 1}\left(P_i(y)\frac{1}{(y+i)^2} + \frac{i}{(y+i)(y+i+1)^3}\right).$$

Because $y \in I = [0,1]$, $y \geq 0$ so

$$\sum_{i\geq 1} \frac{i}{(y+i)(y+i+1)^3} \leq \sum_{i\geq 1} \frac{1}{(i+1)^3} = -1 + \zeta(3).$$

Let $h(u) = u^2$, with which

$$\sum_{i\geq 1} P_i(y) \frac{1}{(y+i)^2} = Uh(y).$$

$h : I \to \mathbb{R}$ is increasing, so Uh is decreasing. Because $P_i(0) = \frac{1}{i(i+1)}$,

$$\sum_{i \geq 1} P_i(y) \frac{1}{(y+i)^2} = Uh(y) \leq Uh(0) = \sum_{i \geq 1} P_i(0) \frac{1}{i^2} = \sum_{i \geq 1} \frac{1}{i^3(i+1)}.$$

Doing partial fractions,

$$\frac{1}{i^3(i+1)} = \frac{1}{i^3} - \frac{1}{i^2} + \frac{1}{i} - \frac{1}{1+i},$$

so

$$\sum_{i \geq 1} \frac{1}{i^3(i+1)} = \zeta(3) - \zeta(2) + 1.$$

Therefore

$$\left| \frac{Uf(y) - Uf(x)}{y - x} \right| \leq s(f) \left(\zeta(3) - \zeta(2) + 1 - 1 + \zeta(3) \right) = s(f)(2\zeta(3) - \zeta(2)).$$

$\square$

For example, let $f(x) = x$, for which $s(f) = 1$. Now,

$$Uf(x) = \sum_{i \geq 1} P_i(x) \frac{1}{x+i}.$$

We remind ourselves that

$$P_i(x) = \frac{x+1}{(x+i)(x+i+1)}, \quad P_i'(x) = \frac{i^2 - i - (x+1)^2}{(x+i)^2(x+i+1)^2}.$$

Then

$$\begin{aligned}
(Uf)'(x) &= \sum_{i \geq 1} \left(P_i'(x) \frac{1}{x+i} - P_i(x) \frac{1}{(x+i)^2} \right) \\
&= \sum_{i \geq 1} \left(\frac{i^2 - i - (x+1)^2}{(x+i)^3(x+i+1)^2} - \frac{x+1}{(x+i)^3(x+i+1)} \right) \\
&= \sum_{i \geq 1} \frac{i^2 - i - (x+1)^2 - (x+1)(x+i+1)}{(x+i)^3(x+i+1)^2} \\
&= \sum_{i \geq 1} \frac{-2x^2 - ix - 4x + i^2 - 2i - 2}{(x+i)^3(x+i+1)^2}.
\end{aligned}$$

Check that $x \mapsto (Uf)'(x)$ is increasing and negative. Then $\|(Uf)'\| \leq |(Uf)'(0)|$, with

$$(Uf)'(0) = \sum_{i \geq 1} \frac{i^2 - 2i - 2}{i^3(i+1)^2} = -2\zeta(3) + \zeta(2).$$

Therefore for $f(x) = x$,

$$s(f) = \|(Uf)'\|_\infty = 2\zeta(3) - \zeta(2),$$

which shows that the above theorem is sharp.