

Locally compact abelian groups

Jordan Bell

jordan.bell@gmail.com

Department of Mathematics, University of Toronto

June 4, 2014

1 Introduction

These notes are a gloss on the first chapter of Walter Rudin's *Fourier Analysis on Groups*, and may be helpful to someone reading Rudin. The results I do prove are proved in more detail than they are in Rudin. I caution that before reading the first chapter of that book it is know about the Gelfand transform on commutative Banach algebras because results from that are used without even stating them by Rudin. I at least state them.

2 Locally compact abelian groups

Let G be a locally compact abelian (LCA) group. There is a Haar measure m on G . It is a fact that if m and m' are Haar measures on G , then there is a positive constant λ such that $m' = \lambda m$.¹ If G is compact then there is a unique Haar measure with $m(G) = 1$, and if G is discrete then there is a unique Haar measure with $m(\{x\}) = 1$ for each $x \in G$.

Because Haar measures on G are positive multiples of each other, the elements of $L^p(G)$ do not depend on the Haar measure we use, but the value of $\|f\|_p$ will, where $f : G \rightarrow \mathbb{C}$ is a Borel function.

If X is a normed vector space and $f : G \rightarrow X$ is a function, we say that f is **uniformly continuous** if for every $\epsilon > 0$ there is a neighborhood V of 0 in G such that $x - y \in V$ implies that $\|f(x) - f(y)\|_X < \epsilon$.

If $f : G \rightarrow \mathbb{C}$ is a function and $x \in G$, we define $f_x : G \rightarrow \mathbb{C}$ by

$$f_x(y) = f(y - x), \quad y \in G.$$

Theorem 1. *If $1 \leq p < \infty$ and $f \in L^p(G)$, then*

$$x \mapsto f_x$$

is uniformly continuous $G \rightarrow L^p(G)$.

¹Walter Rudin, *Fourier Analysis on Groups*, p. 2, §1.1.3.

Proof. Let $\epsilon > 0$. Because $C_c(G)$ is dense in $L^p(G)$, there is some $g \in C_c(G)$ such that $\|g - f\|_p < \epsilon$. Let $K = \text{supp } g$, and because K is compact, $m(K) < \infty$. $g \in C_c(G)$ implies that g is uniformly continuous on G , so there is a neighborhood V of 0 in G such that if $x - y \in V$ then $|g(x) - g(y)| < \epsilon(2m(K))^{-1/p}$. Then, for all $x \in V$ and $y \in G$, as $y - (y - x) \in V$ we have $|g(y - x) - g(y)| < \epsilon(2m(K))^{-1/p}$. That is, for all $x \in V$ we have

$$\|g_x - g\|_\infty < \epsilon(2m(K))^{-1/p}.$$

For $x \in V$, $\text{supp}(g_x - g) \subset (K + x) \cup K$ and $m(K + x) = m(K)$, so for all $x \in V$,

$$\begin{aligned} \|g_x - g\|_p &= \left(\int_{\text{supp}(g_x - g)} |g_x - g|^p dm \right)^{1/p} \\ &\leq \left(\int_{\text{supp}(g_x - g)} \epsilon^p (2m(K))^{-1} dm \right)^{1/p} \\ &\leq \epsilon. \end{aligned}$$

Because $\|f_x - g_x\|_p = \|f - g\|_p$, for all $x \in V$ we have

$$\|f_x - f\|_p \leq \|f_x - g_x\|_p + \|g_x - g\|_p + \|g - f\|_p < 3\epsilon.$$

Then, let $x, y \in G$ with $y - x \in V$. The above inequality tells us

$$\|f_{y-x} - f\|_p < 3\epsilon.$$

But $f_x - f_y = (f - f_{y-x})_x$ and $\|h_x\|_p = \|h\|_p$, so

$$\|f_x - f_y\|_p = \|f - f_{y-x}\|_p < 3\epsilon,$$

showing that $x \mapsto f_x$ is uniformly continuous. \square

If f and g are Borel functions on G , for $x \in G$ such that the integral exists we define

$$(f * g)(x) = \int_G f(x - y)g(y)dm(y).$$

The operation of taking the convolution of functions is particularly suitable for functions that belong to $L^1(G)$, because if $f, g \in L^1(G)$, then $(f * g)(x)$ is defined for almost all $x \in G$, and satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1;$$

this is proved in Rudin, together with other properties of the convolution.² From the above inequality, it follows that $L^1(G)$ with convolution as multiplication is a commutative Banach algebra. The map $*$: $L^1(G) \rightarrow L^1(G)$ defined by $f^*(x) = \overline{f(-x)}$ for $f \in L^1(G)$, $x \in G$, is an isometric involution.

²Walter Rudin, *Fourier Analysis on Groups*, p. 4, §1.1.6.

If G is discrete, define e on G by $e(0) = 1$ and $e(x) = 0$ for $x \neq 0$. Then $\|e\|_1 = \int_G e(x) dm(x) = \sum_{x \in G} e(x) = 1$, so $e \in L^1(G)$. For $f \in L^1(G)$ and $x \in G$,

$$(f * e)(x) = \sum_{y \in G} f(x - y)e(y) = f(x)e(0) = f(x),$$

showing that $f * e = f$. Hence e is unity in $L^1(G)$. It turns out that if G is not discrete then $L^1(G)$ does not have a unity.³

3 Dual groups

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with the subspace topology inherited from \mathbb{C} . If G is a locally compact abelian group, we denote by Γ the set of continuous homomorphisms $G \rightarrow \mathbb{T}$. For $\gamma \in \Gamma$ and $x \in G$, we write

$$\langle x, \gamma \rangle = \gamma(x).$$

We call a homomorphism $G \rightarrow \mathbb{T}$ a **character** of G , so elements of Γ are the continuous characters of G .

If A is a Banach algebra and Δ is the set of algebra homomorphisms $h : A \rightarrow \mathbb{C}$ that are not identically zero, the **Gelfand transform** of $x \in A$ is the map $\hat{x} : \Delta \rightarrow \mathbb{C}$ defined by

$$\hat{x}(h) = h(x), \quad h \in \Delta.$$

Δ is called the **maximal ideal space** of A .

If $f \in L^1(G)$, we define $\hat{f} : \Gamma \rightarrow \mathbb{C}$ by

$$\hat{f}(\gamma) = \int_G f(x) \langle -x, \gamma \rangle dm(x), \quad \gamma \in \Gamma,$$

and call \hat{f} the **Fourier transform** of f .

The following theorem is from Rudin.⁴ It establishes a bijection between the dual group Γ of G and the maximal ideal space Δ of $L^1(G)$, and shows, if Γ and Δ are identified, that the Fourier transform is the same as the Gelfand transform.

Theorem 2. *For each $\gamma \in \Gamma$, the map $h : L^1(G) \rightarrow \mathbb{C}$ defined by $h(f) = \hat{f}(\gamma)$ belongs to Δ . If $h \in \Delta$, then there is a unique $\gamma \in \Gamma$ such that $h(f) = \hat{f}(\gamma)$ for all $f \in L^1(G)$.*

Proof. For $f, g \in L^1(G)$ and $k = f * g$, then for any $\gamma \in \Gamma$, using $\langle -x, \gamma \rangle =$

³Walter Rudin, *Fourier Analysis on Groups*, p. 30, §1.7.3.

⁴Walter Rudin, *Fourier Analysis on Groups*, p. 7, §1.2.2.

$$\langle -x + y, \gamma \rangle \langle -y, \gamma \rangle,$$

$$\begin{aligned} \hat{k}(\gamma) &= \int_G (f * g)(x) \langle -x, \gamma \rangle dm(x) \\ &= \int_G \langle -x, \gamma \rangle \left(\int_G f(x - y) g(y) dm(y) \right) dm(x) \\ &= \int_G g(y) \langle -y, \gamma \rangle \left(\int_G f(x - y) \langle -x + y, \gamma \rangle dm(x) \right) dm(y) \\ &= \int_G g(y) \langle -y, \gamma \rangle \hat{f}(x) dm(y) \\ &= \hat{f}(\gamma) \hat{g}(\gamma), \end{aligned}$$

showing that $f \mapsto \hat{f}(\gamma)$ is an algebra homomorphism. By Urysohn's lemma, for every neighborhood V of $0 \in G$, there is some $f \in C_c(G)$ with $f(0) = 1$, $0 \leq f \leq 1$, and $\text{supp } f \subset V$. As $\langle 0, \gamma \rangle = 1$ and because $x \mapsto \langle -x, \gamma \rangle$ is continuous, it follows that there is some neighborhood V of 0 and some f as above such that $\hat{f}(\gamma) = \int_G f(x) \langle -x, \gamma \rangle dm(x) \neq 0$, showing that $f \mapsto \hat{f}(\gamma)$ is nonzero.

Now let $h : L^1(G) \rightarrow \mathbb{C}$ be a nonzero algebra homomorphism. A homomorphism from a Banach algebra to \mathbb{C} is linear functional with norm 1. It is a fact that for every bounded linear functional Λ on $L^1(G)$ there is some $\phi \in L^\infty(G)$ such that $\Lambda(f) = \int_G f \phi dm$ for all $f \in L^1(G)$, and $\|\phi\|_\infty = \|\Lambda\|$.⁵ Therefore, there is some $\phi \in L^\infty(G)$, $\|\phi\|_\infty = 1$, such that

$$h(f) = \int_G f \phi dm, \quad f \in L^1(G).$$

Then for $f, g \in L^1(G)$ we have

$$\begin{aligned} h(f) \int_G g \phi dm &= h(f) h(g) \\ &= h(f * g) \\ &= \int_G (f * g) \phi dm \\ &= \int_G \left(\int_G f(x - y) g(y) dm(y) \right) \phi(x) dm(x) \\ &= \int_G g(y) \left(\int_G f(x - y) \phi(x) dm(x) \right) dm(y) \\ &= \int_G g(y) h(f_y) dm(y), \end{aligned}$$

i.e.

$$\int_G (h(f) \phi(y) - h(f_y)) g(y) dm(y) = 0.$$

⁵Walter Rudin, *Fourier Analysis on Groups*, p. 268, E10.

Because this is true for all $g \in L^1(G)$, it follows that for all $f \in L^1(G)$ and for almost all $y \in G$,

$$h(f)\phi(y) = h(f_y).$$

Because $h \neq 0$, there is some $g \in L^1(G)$ with $h(g) \neq 0$. Then for almost all $(x, y) \in G \times G$,

$$h(g)\phi(x+y) = h(g_{x+y}) = h((g_x)_y) = h(g_x)\phi(y) = h(g)\phi(x)\phi(y),$$

and as $h(g) \neq 0$, for almost all $(x, y) \in G \times G$,

$$\phi(x+y) = \phi(x)\phi(y).$$

We define $\phi_1 : G \rightarrow \mathbb{C}$ by $\phi_1(y) = \frac{h(g_y)}{h(g)}$, which is continuous because $y \mapsto g_y$ is continuous and h is continuous. The function ϕ_1 satisfies $\phi(y) = \phi_1(y)$ for almost all $y \in G$, so for almost all $(x, y) \in G \times G$,

$$\phi_1(x+y) = \phi_1(x)\phi_1(y).$$

$(x, y) \mapsto \phi_1(x+y)$ and $(x, y) \mapsto \phi_1(x)\phi_1(y)$ are continuous and the above equality holds for almost all $(x, y) \in G \times G$, so the above equality in fact holds for all $(x, y) \in G \times G$. Hence $\phi_1 \in \Gamma$. Define $\gamma : G \rightarrow \mathbb{C}$ by $\gamma(x) = \phi_1(-x)$. Then $\gamma \in \Gamma$, and $\gamma(x) = \phi(-x)$ for almost all $x \in G$, so for all $f \in L^1(G)$,

$$\begin{aligned} \hat{f}(\gamma) &= \int_G f(x)\langle -x, \gamma \rangle dm(x) \\ &= \int_G f(x)\gamma(-x)dm(x) \\ &= \int_G f(x)\phi(x)dm(x) \\ &= h(f). \end{aligned}$$

Suppose that $\gamma_1, \gamma_2 \in \Gamma$ satisfy $\hat{f}(\gamma_1) = \hat{f}(\gamma_2)$ for all $f \in L^1(G)$. Then $\langle -x, \gamma_1 \rangle = \langle -x, \gamma_2 \rangle$ for almost all $x \in G$, and because these are continuous they are in fact equal for all $x \in G$, i.e. $\gamma_1 = \gamma_2$. \square

Let $A(\Gamma) = \{\hat{f} : f \in L^1(G)\}$. Elements of $A(\Gamma)$ are functions $\Gamma \rightarrow \mathbb{C}$. So far Γ has not been given a topology. We shall be interested in the initial topology on Γ with respect to the family of functions $A(\Gamma)$. That is, the topology on Γ is the coarsest topology so that each element of $A(\Gamma)$ is continuous. Furthermore, it is a fact that the topology on Γ is equal to the subspace topology on Γ inherited from $L^1(G)^*$ with the weak-* topology. Finally, let the maximal ideal space Δ of $L^1(G)$ have the subspace topology inherited from $L^1(G)^*$ with the weak-* topology. In Theorem 2 we presented a bijection $\Gamma \rightarrow \Delta$, and can be proved that this bijection is a homeomorphism.

It is proved in Rudin⁶ that with this topology, Γ is a locally compact abelian group. The proof constructs a basis for the topology of Γ , and because we will use this basis later it will be useful to write out the proof.

⁶Walter Rudin, *Fourier Analysis on Groups*, p. 10, §1.2.6.

Theorem 3. 1. $(x, \gamma) \mapsto \langle x, \gamma \rangle$ is continuous $G \times \Gamma \rightarrow \mathbb{C}$.

2. For $r > 0$ let $U_r = \{z \in \mathbb{C} : |1 - z| < r\}$. If K is a compact subset of G then

$$W(K, r) = \{\gamma \in \Gamma : \langle x, \gamma \rangle \in U_r \text{ for all } x \in K\}$$

is an open subset of Γ , and if C is a compact subset of Γ then

$$V(C, r) = \{x \in G : \langle x, \gamma \rangle \in U_r \text{ for all } \gamma \in C\}$$

is an open subset of G .

3. $\{\gamma_0 + W(K, r) : \gamma_0 \in \Gamma, K \text{ is a compact subset of } \Gamma, r > 0\}$ is a basis for the topology of Γ .

4. Γ is a locally compact abelian group.

Proof. Let $f \in L^1(G)$. For $x_0 \in G$, $\gamma_0 \in \Gamma$ and $\epsilon > 0$, because $x \mapsto f_x$ is continuous there is a neighborhood V of x_0 such that $\|f_x - f_{x_0}\|_1 < \epsilon$ for all $x \in V$, and because \hat{f}_{x_0} is continuous there is a neighborhood W of γ_0 such that $|\hat{f}_{x_0}(\gamma) - \hat{f}_{x_0}(\gamma_0)| < \epsilon$ for all $\gamma \in W$. If $(x, \gamma) \in V \times W$, then

$$\begin{aligned} |\hat{f}_x(\gamma) - \hat{f}_{x_0}(\gamma_0)| &\leq |\hat{f}_x(\gamma) - \hat{f}_{x_0}(\gamma)| + |\hat{f}_{x_0}(\gamma) - \hat{f}_{x_0}(\gamma_0)| \\ &< \|f_x - f_{x_0}\|_1 + \epsilon \\ &< 2\epsilon, \end{aligned}$$

showing that $(x, \gamma) \mapsto \hat{f}_x(\gamma)$ is continuous. But

$$\langle x, \gamma \rangle \hat{f}(\gamma) = \hat{f}_x(\gamma), \quad x \in G, \gamma \in \Gamma,$$

and it follows that $(x, \gamma) \mapsto \langle x, \gamma \rangle$ is continuous.

Let K be a compact subset of G , $r > 0$, and $\gamma_0 \in W(K, r)$. For $x_0 \in K$, $|\langle x_0, \gamma_0 \rangle - 1| = \delta < r$. Because $(x, \gamma) \mapsto \langle x, \gamma \rangle$ is continuous there is a neighborhood V_{x_0} of x_0 and a neighborhood W_{x_0} of γ_0 such that $|\langle x, \gamma \rangle - \langle x_0, \gamma_0 \rangle| < r - \delta$ for $(x, \gamma) \in V_{x_0} \times W_{x_0}$, and then $\langle x, \gamma \rangle \in U_r$ for $(x, \gamma) \in V_{x_0} \times W_{x_0}$. As $K \subset \bigcup_{x_0 \in K} V_{x_0}$ and K is compact, there are $x_1, \dots, x_n \in K$ such that $K \subset \bigcup V_{x_i}$. Let $W = \bigcap W_{x_i}$, which is a finite intersection of neighborhoods of γ_0 and hence itself a neighborhood of γ_0 . It is apparent that $W \subset W(K, r)$, showing that $W(K, r)$ is open.

Let C be a compact subset of G , $r > 0$, and $x_0 \in V(C, r)$. For $\gamma_0 \in C$, $|\langle x_0, \gamma_0 \rangle - 1| = \delta < r$, so there is a neighborhood V_{γ_0} of x_0 and a neighborhood W_{γ_0} of γ_0 such that $|\langle x, \gamma \rangle - \langle x_0, \gamma_0 \rangle| < r - \delta$ for $(x, \gamma) \in V_{\gamma_0} \times W_{\gamma_0}$, therefore $\langle x, \gamma \rangle \in U_r$ for $(x, \gamma) \in V_{\gamma_0} \times W_{\gamma_0}$. There are $\gamma_1, \dots, \gamma_n \in C$ such that $C \subset \bigcup W_{\gamma_i}$, and $V = \bigcap V_{\gamma_i}$ is a neighborhood of x_0 that is contained in $V(C, r)$, showing that $V(C, r)$ is open.

Let $\gamma_0 \in \Gamma$ and let W be a neighborhood of γ_0 . Because Γ has the initial topology for $A(\Gamma)$, a local subbasis at γ_0 is given by sets of the form $\{\gamma \in \Gamma :$

$|\hat{f}(\gamma) - \hat{f}(\gamma_0)| < \epsilon\}$, $f \in L^1(G)$ and $\epsilon > 0$. Therefore, there are $f_1, \dots, f_n \in L^1(G)$ and $\epsilon_1, \dots, \epsilon_n > 0$ such that

$$\bigcap \{\gamma \in \Gamma : |\hat{f}_i(\gamma) - \hat{f}_i(\gamma_0)| < \epsilon_i\} \subset W. \quad (1)$$

Let ϵ be the minimum of the ϵ_i , let $g_i \in C_0(G)$ with $\|f_i - g_i\|_1 < \frac{\epsilon}{3}$, let K be the union of the supports of g_i , and let M be the maximum of $\|g_i\|_1$. With $r < \frac{\epsilon}{3M}$, for $\gamma \in \gamma_0 + W(K, r)$ we have

$$\begin{aligned} |\hat{f}_i(\gamma) - \hat{f}_i(\gamma_0)| &\leq |\hat{f}_i(\gamma) - \hat{g}_i(\gamma)| + |\hat{g}_i(\gamma) - \hat{g}_i(\gamma_0)| + |\hat{g}_i(\gamma_0) - \hat{f}_i(\gamma_0)| \\ &\leq \|f_i - g_i\|_1 + \left| \int_K g_i(x) (\langle -x, \gamma \rangle - \langle -x, \gamma_0 \rangle) dm(x) \right| + \|f_i - g_i\|_1 \\ &< \frac{2}{3}\epsilon + \int_K |g_i(x)| |1 - \langle -x, \gamma - \gamma_0 \rangle| dm(x) \\ &\leq \frac{2}{3}\epsilon + r \|g_i\|_1 \\ &< \epsilon. \end{aligned}$$

Thus, if $\gamma \in \gamma_0 + W(K, r)$ then γ belongs to the intersection (1), and this shows that $\gamma_0 + W(K, r) \subset W$. This establishes that the collection of those sets of the form $\gamma_0 + W(K, r)$, for $\gamma_0 \in \Gamma$, K a compact subset of Γ , and $r > 0$, is a basis for the topology of Γ .

Let $\gamma_1, \gamma_2 \in \Gamma$, let K be a compact subset of Γ , and let $r > 0$. Because

$$(\gamma_1 + W(K, r/2)) - (\gamma_2 + W(K, r/2)) \subset \gamma_1 - \gamma_2 + W(K, r),$$

for $(\gamma', \gamma'') \in (\gamma_1 + W(K, r/2)) \times (\gamma_2 + W(K, r/2))$ we have $\gamma' - \gamma'' \in \gamma_1 - \gamma_2 + W(K, r)$, and this shows that $(\gamma', \gamma'') \mapsto \gamma' - \gamma''$ is continuous, which shows that Γ is a topological group. Γ is locally compact because it is homeomorphic to the maximal ideal space Δ of $L^1(G)$, and it is a fact that the maximal ideal space of any commutative Banach algebra is a locally compact Hausdorff space. This completes the proof. \square

A fact that Rudin uses but merely asserts is the following (he refers to it being true for the Gelfand transform, and then merely asserts its truth there).

Theorem 4. $A(\Gamma) \subset C_0(\Gamma)$.

Proof. Let Γ' be the weak-* closure of Γ in $L^1(G)^*$. For $\Lambda \in \Gamma'$, there is a net $\gamma_i \in \Gamma$ that weak-* converges to Λ . For $x, y \in G$,

$$\Lambda(ab) = \lim_i \gamma_i(ab) = \lim_i \gamma_i(a)\gamma_i(b) = \left(\lim_i \gamma_i(a) \right) \left(\lim_i \gamma_i(b) \right) = \Lambda(a)\Lambda(b).$$

Similarly, Λ is linear, so Λ is an algebra homomorphism $G \rightarrow \mathbb{C}$. Hence taking the closure of Γ in $L^1(G)^*$ has added precisely the map that is identically 0:

$$\Gamma' = \Gamma \cup \{0\}.$$

With $K = \{\Lambda \in L^1(G)^* : \|\Lambda\| \leq 1\}$, the Banach-Alaoglu theorem tells us that K is a weak-* compact subset of $L^1(G)^*$. It is apparent that $\Gamma' \subset K$, so Γ' is a weak-* compact subset of $L^1(G)^*$.

If $f \in L^1(G)$ and $\epsilon > 0$, then because $\hat{f} : \Gamma \rightarrow \mathbb{C}$ is continuous,

$$K_0 = \{\gamma \in \Gamma : |\hat{f}(\gamma)| \geq \epsilon\}$$

is a closed subset of Γ . The only way K_0 would fail to be a weak-* closed subset of Γ' is if 0 belonged to its weak-* closure, and it is straightforward to see that this is not the case. So K_0 is in fact a weak-* closed subset of the weak-* compact set Γ' , and hence is itself weak-* compact. It follows that K_0 is a compact subset of Γ , and this shows that $\hat{f} \in C_0(\Gamma)$. \square

Now that we know that $A(\Gamma) \subset C_0(\Gamma)$, it does not take long to verify that the conditions of the Stone-Weierstrass theorem are satisfied (for distinct $\gamma_1, \gamma_2 \in \Gamma$ there is some $f \in L^1(G)$ with $\hat{f}(\gamma_1) \neq \hat{f}(\gamma_2)$; $A(\Gamma)$ is self-adjoint; for each $\gamma \in \Gamma$ there is some $f \in L^1(G)$ with $\hat{f}(\gamma) \neq 0$), and hence that $A(\Gamma)$ is dense in $C_0(\Gamma)$.

Theorem 5. *If G is discrete then Γ is compact, and if G is compact then Γ is discrete.*

Proof. We remarked earlier that the bijection $\Gamma \rightarrow \Delta$ is a homeomorphism, where Δ is the maximal ideal space of $L^1(G)$ and has the subspace topology inherited from $L^1(G)^*$ with the weak-* topology. If G is discrete, then $L^1(G)$ is unital, and it is a fact that the maximal ideal space of a unital commutative Banach algebra is compact, so Γ is compact.

Suppose that G is compact, with Haar measure m satisfying $m(G) = 1$. If $\gamma \in \Gamma$ and there is some $x_0 \in G$ with $\gamma(x_0) \neq 1$ (i.e. γ is not the 0 homomorphism), then

$$\int_G \langle x, \gamma \rangle dm(x) = \langle x_0, \gamma \rangle \int_G \langle x - x_0, \gamma \rangle dm(x) = \langle x_0, \gamma \rangle \int_G \langle x, \gamma \rangle dm(x).$$

As $\langle x_0, \gamma \rangle \neq 1$, this means that $\int_G \langle x, \gamma \rangle dm(x) = 0$. Therefore,

$$\int_G \langle x, \gamma \rangle dm(x) = \begin{cases} 1 & \gamma = 0, \\ 0 & \gamma \neq 0. \end{cases}$$

As G is compact, $\chi_G \in L^1(G)$, and

$$\hat{\chi}_G(\gamma) = \int_G \chi_G(x) \langle -x, \gamma \rangle dm(x) = \int_G \langle -x, \gamma \rangle dm(x) = \begin{cases} 1 & \gamma = 0, \\ 0 & \gamma \neq 0. \end{cases}$$

$\hat{\chi}_G$ is continuous, so $\{\gamma \in \Gamma : \hat{\chi}_G(\gamma) = 0\}$ is a closed subset of Γ , hence its complement $\{\gamma \in \Gamma : \hat{\chi}_G(\gamma) \neq 0\}$ is an open subset of Γ . But by the above this complement is $\{0\}$, and Γ is a topological group so this implies that each singleton is open, meaning that Γ is discrete. \square

4 Regular complex Borel measures on G

Let $M(G)$ be the set of regular complex Borel measures on G . If E is a Borel set in G , define $E_n = \{(x_1, \dots, x_n) : x_1 + \dots + x_n \in E\}$, which is a Borel set in G^n . For $\mu_1, \dots, \mu_n \in M(G)$, we define

$$(\mu_1 * \dots * \mu_n)(E) = (\mu_1 \times \dots \times \mu_n)(E_n).$$

It is proved in Rudin⁷ that $\mu_1 * \dots * \mu_n \in M(G)$, and that with convolution as multiplication and norm $\|\mu\| = |\mu|(G)$ (the **total variation norm**), $M(G)$ is a commutative Banach algebra with unity δ_0 .

The **Fourier transform** of $\mu \in M(G)$ is the function $\hat{\mu} : \Gamma \rightarrow \mathbb{C}$ defined by

$$\hat{\mu}(\gamma) = \int_G \langle -x, \gamma \rangle d\mu(x), \quad \gamma \in \Gamma.$$

We write $B(\Gamma) = \{\hat{\mu} : \mu \in M(G)\}$. For $f \in L^1(G)$ we have proved that $\hat{f} \in C_0(\Gamma)$. We prove now that for $\mu \in M(G)$, $\hat{\mu}$ is bounded and uniformly continuous on Γ . However, $\delta_0 \in M(G)$, and for $\gamma \in \Gamma$,

$$\hat{\delta}_0(\gamma) = \int_G \langle -x, \gamma \rangle d\delta_0(x) = \langle 0, \gamma \rangle = \gamma(0) = 1,$$

so $\hat{\delta}_0 \notin C_0(\Gamma)$. The proof is from Rudin.⁸

Theorem 6. *If $\mu \in M(G)$, then $\hat{\mu} : \Gamma \rightarrow \mathbb{C}$ is bounded and uniformly continuous.*

Proof. For any $\gamma \in \Gamma$,

$$|\hat{\mu}(\gamma)| \leq \int_G |\langle -x, \gamma \rangle| d|\mu|(x) = \int_G d|\mu|(x) = |\mu|(G) < \infty,$$

where $|\mu|$ is the variation of μ . Hence $\hat{\mu}$ is bounded.

$|\mu|$ is regular, so for any $\delta > 0$ there is some compact set K such that $|\mu|(K') < \delta$, where $K' = G \setminus K$. For $\gamma_1, \gamma_2 \in \Gamma$, $\langle x, \gamma_1 - \gamma_2 \rangle = \langle x, \gamma_1 \rangle \langle x, \gamma_2 \rangle^{-1}$ and hence

$$|\langle x, \gamma_1 \rangle - \langle x, \gamma_2 \rangle| = |1 - \langle x, \gamma_1 - \gamma_2 \rangle|,$$

with which we get

$$|\hat{\mu}(\gamma_1) - \hat{\mu}(\gamma_2)| \leq \int_G |1 - \langle x, \gamma_1 - \gamma_2 \rangle| d|\mu|(x).$$

We know that $W(K, \delta)$ defined in Theorem 3 is an open neighborhood of 0. If $\gamma_1 - \gamma_2 \in W(K, \delta)$, then by the definition of $W(K, \delta)$, for all $x \in K$ we have $|1 - \langle x, \gamma_1 - \gamma_2 \rangle| < \delta$, giving

$$\int_G |1 - \langle x, \gamma_1 - \gamma_2 \rangle| d|\mu|(x) \leq \int_K \delta d|\mu|(x) + \int_{K'} 2d|\mu|(x) < \delta\|\mu\| + 2\delta,$$

⁷Walter Rudin, *Fourier Analysis on Groups*, p. 13, §1.3.1.

⁸Walter Rudin, *Fourier Analysis on Groups*, p. 15, §1.3.3.

showing that $\hat{\mu}$ is uniformly continuous. □

If $f \in L^1(G)$, we define $\mu_f \in M(G)$ by

$$\mu_f(E) = \int_E f(x) dm(x),$$

for Borel subsets E of G , where m is Haar measure on G . Thus, μ_f is absolutely continuous with respect to m and has density f . Then, for $\gamma \in \Gamma$,

$$\hat{\mu}_f(\gamma) = \int_G \langle -x, \gamma \rangle d\mu_f(x) = \int_G \langle -x, \gamma \rangle f(x) dm(x) = \hat{f}(\gamma),$$

showing that $A(\Gamma) \subset B(\Gamma)$. Furthermore, $\|f\|_1 = \|\mu_f\|$, hence it makes sense to identify $L^1(G)$ with its image in $M(G)$ under the map $f \mapsto \mu_f$. To talk about a function that belongs to $M(G)$ is to speak about μ_f for some $f \in L^1(G)$. It is proved in Rudin that $L^1(G)$ is a closed ideal in the Banach algebra $M(G)$.⁹

Rudin calls the following theorem the **uniqueness theorem**.¹⁰

Theorem 7. *If $\mu \in M(\Gamma)$ and*

$$\int_{\Gamma} \langle x, \gamma \rangle d\mu(\gamma) = 0$$

for all $x \in G$, then $\mu = 0$.

Proof. Let $f \in L^1(G)$.

$$\begin{aligned} \int_{\Gamma} \hat{f}(\gamma) d\mu(\gamma) &= \int_{\Gamma} \int_G f(x) \langle -x, \gamma \rangle dm(x) d\mu(\gamma) \\ &= \int_G f(x) \int_{\Gamma} \langle -x, \gamma \rangle d\mu(\gamma) dm(x) \\ &= 0. \end{aligned}$$

Because $A(\Gamma)$ is dense in $C_0(\Gamma)$, it follows that for all $\phi \in C_0(\Gamma)$,

$$\int_{\Gamma} \phi d\mu = 0,$$

and this implies that $\mu = 0$. □

5 Positive-definite functions

A function $\phi : G \rightarrow \mathbb{C}$ is called **positive-definite** if for every N and every $x_1, \dots, x_N \in G$, $c_1, \dots, c_N \in \mathbb{C}$, we have

$$\sum_{n,m=1}^N c_n \overline{c_m} \phi(x_n - x_m) \geq 0. \quad (2)$$

⁹Walter Rudin, *Fourier Analysis on Groups*, p. 16, §1.3.4.

¹⁰Walter Rudin, *Fourier Analysis on Groups*, p. 17, §1.3.6.

In particular, the left-hand side of the above inequality is real.

Let ϕ be a character of G , which we do not assume to be continuous. Then,

$$\sum_{n,m=1}^N c_n \overline{c_m} \phi(x_n - x_m) = \sum_{n,m=1}^N c_n \overline{c_m} \phi(x_n) \overline{\phi(x_m)} = \left| \sum_{n=1}^N c_n \phi(x_n) \right|^2 \geq 0,$$

so any character of G , whether or not it is continuous, is positive-definite.

Lemma 8. *If $\phi : G \rightarrow \mathbb{C}$ is positive-definite, then*

$$\phi(0) \geq 0,$$

$$\phi(-x) = \overline{\phi(x)}, \quad |\phi(x)| \leq \phi(0), \quad x \in G,$$

and

$$|\phi(x) - \phi(y)|^2 \leq 2\phi(0)\operatorname{Re}(\phi(0) - \phi(x - y)), \quad x, y \in G.$$

Proof. Take $N = 1$, $c_1 = 1$, $x_1 = 0$. Then (2) is

$$\phi(0) \geq 0.$$

Take $N = 2$, $x_1 = 0$, $x_2 = x$, $c_1 = 1$, $c_2 = c$. Then (2) is

$$\phi(0) + \overline{c}\phi(-x) + c\phi(x) + |c|^2\phi(0) \geq 0. \quad (3)$$

Because $\phi(0)$ is real, this means that $c\phi(x) + \overline{c}\phi(-x)$ is real, hence is equal to its own complex conjugate. Writing $\phi(x) = A + iB$ and $\phi(-x) = C + iD$, for $c = 1$ this implies that

$$A + iB + C + iD = A - iB + C - iD$$

and for $c = i$ this is

$$iA - B - iC + D = -iA - B + iC + D.$$

The first equation tells us $B + D = 0$, and the second equation tells us $A - C = 0$.

Thus

$$\phi(-x) = C + iD = A - iB = \overline{\phi(x)}.$$

Use (3) with c chosen so that $c\phi(x) = -|\phi(x)|$. $|c| = 1$, and using $\phi(-x) = \overline{\phi(x)}$,

$$2\phi(0) + 2|\phi(x)| \geq 0.$$

Take $N = 3$, $x_1 = 0$, $x_2 = x$, $x_3 = y$, $c_1 = 1$, $\lambda \in \mathbb{R}$,

$$c_2 = \frac{\lambda|\phi(x) - \phi(y)|}{\phi(x) - \phi(y)},$$

$c_3 = -c_2$; since $\phi(0) \geq 0$, the claim is obviously true for the case $x = y$, which we discard. Then (2) is

$$(1+2|c_2|^2)\phi(0)+\overline{c_2}(\phi(-x)-\phi(-y))+c_2(\phi(x)-\phi(y))-|c_2|^2(\phi(x-y)+\phi(y-x)) \geq 0.$$

Using the definition of c_2 and the fact that $\overline{\phi(z)} = \phi(-z)$,

$$(1+2\lambda^2)\phi(0)+2\lambda|\phi(x)-\phi(y)|-\lambda^2(\phi(x-y)+\overline{\phi(x-y)}) \geq 0,$$

or

$$\lambda^2(2\phi(0)-2\operatorname{Re}\phi(x-y))+2\lambda|\phi(x)-\phi(y)|+\phi(0) \geq 0.$$

The fact that this quadratic polynomial does not take negative values implies that it has either 0 or 1 real roots, and hence that its discriminant is ≤ 0 :

$$4|\phi(x)-\phi(y)|^2-4(2\phi(0)-2\operatorname{Re}\phi(x-y))\phi(0) \leq 0,$$

which is the claim. \square

We remind ourselves that $f^*(x) = \overline{f(-x)}$.

Theorem 9. *If $f \in L^2(G)$, then $\phi = f * f^*$ is positive-definite and belongs to $C_0(G)$.*

Proof.

$$\begin{aligned} \sum_{n,m=1}^N c_n \overline{c_m} \phi(x_n - x_m) &= \sum_{n,m=1}^N c_n \overline{c_m} \int_G f(x_n - x_m - y) \overline{f(-y)} dy \\ &= \sum_{n,m=1}^N c_n \overline{c_m} \int_G f(x_n - y) \overline{f(x_m - y)} dy \\ &= \int_G \left| \sum_{n=1}^N c_n f(x_n - y) \right|^2 dy \\ &\geq 0. \end{aligned}$$

It is a fact that if $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p(G)$, $g \in L^q(G)$, then $f * g \in C_0(G)$.¹¹ \square

The following theorem is from Rudin.¹²

Theorem 10. *If $\mu \in M(\Gamma)$, $\mu \geq 0$ (i.e. $\mu = |\mu|$), and*

$$\phi(x) = \int_{\Gamma} \langle x, \gamma \rangle d\mu(\gamma), \quad x \in G,$$

then ϕ is uniformly continuous and positive-definite.

¹¹Walter Rudin, *Fourier Analysis on Groups*, p. 4, §1.1.6.

¹²Walter Rudin, *Fourier Analysis on Groups*, p. 19, §1.4.2.

Proof.

$$\begin{aligned}
\sum_{n,m=1}^N c_n \overline{c_m} \phi(x_n - x_m) &= \int_{\Gamma} \sum_{n,m=1}^N c_n \overline{c_m} \langle x_n - x_m, \gamma \rangle d\mu(\gamma) \\
&= \int_{\Gamma} \sum_{n,m=1}^N c_n \overline{c_m} \langle x_n, \gamma \rangle \overline{\langle x_m, \gamma \rangle} d\mu(\gamma) \\
&= \int_{\Gamma} \left| \sum_{n=1}^N c_n \langle x_n, \gamma \rangle \right|^2 d\mu(\gamma) \\
&\geq 0,
\end{aligned}$$

showing that ϕ is positive-definite.

μ is regular, so for any $\delta > 0$ there is a compact set C such that $\mu(C') < \delta$, where $C' = \Gamma \setminus C$. We know $V(C, \delta)$ defined in Theorem 3 is an open neighborhood of 0. If $x_1 - x_2 \in V(C, \delta)$, then by the definition of $V(C, \delta)$, for all $\gamma \in C$ we have $|1 - \langle x_1 - x_2, \gamma \rangle| < \delta$, and so

$$\begin{aligned}
|\phi(x_1) - \phi(x_2)| &\leq \int_{\Gamma} |1 - \langle x_1 - x_2, \gamma \rangle| d\mu(\gamma) \\
&= \int_C |1 - \langle x_1 - x_2, \gamma \rangle| d\mu(\gamma) + \int_{C'} |1 - \langle x_1 - x_2, \gamma \rangle| d\mu(\gamma) \\
&\leq \int_C \delta d\mu(\gamma) + \int_{C'} 2 d\mu(\gamma) \\
&< \delta \|\mu\| + 2\delta.
\end{aligned}$$

□

The above theorem shows that the inverse Fourier transform of a nonnegative measure on Γ is a uniformly continuous positive-definite function on G . The following theorem shows that any continuous positive-definite function on G has this form. We remind ourselves that if a positive-definite function is continuous then it is uniformly continuous, by Lemma 8. We are following the proof given in Rudin.¹³

Theorem 11 (Bochner's theorem). *If $\phi : G \rightarrow \mathbb{C}$ is uniformly continuous and positive-definite, then there is some nonnegative measure $\mu \in M(\Gamma)$ such that*

$$\phi(x) = \int_{\Gamma} \langle x, \gamma \rangle d\mu(\gamma), \quad x \in G.$$

Proof. As ϕ is positive-definite, $\phi(0)$ is a nonnegative real number, and $|\phi(x)| \leq \phi(0)$ for all $x \in G$. If $\phi(0) = 0$ then use $\mu = 0$. Otherwise, it makes sense to divide ϕ by $\phi(0)$ and the resulting function is also continuous and positive-definite. Thus without loss of generality we suppose that $\phi(0) = 1$.

¹³Walter Rudin, *Fourier Analysis on Groups*, p. 19, §1.4.3.

Using the fact that ϕ is positive-definite one shows that for any $f \in C_c(G)$, $\int_G \int_G f(x)\overline{f(y)}\phi(x-y)dm(x)dm(y)$ is nonnegative by approximating this integral with finite sums. Then as $C_c(G)$ is dense in $L^1(G)$, the previous integral is nonnegative for any $f \in L^1(G)$. We define $T_\phi : L^1(G) \rightarrow \mathbb{C}$ by

$$T_\phi(f) = \int_G f\phi dm, \quad f \in L^1(G),$$

and define, for $f, g \in L^1(G)$,

$$\begin{aligned} [f, g] &= T_\phi(f * g^*) \\ &= \int_G (f * g^*)(x)\phi(x)dm(x) \\ &= \int_G \int_G f(x-y)\overline{g(-y)}\phi(x)dm(y)dm(x) \\ &= \int_G \int_G f(x)\overline{g(y)}\phi(x-y)dm(x)dm(y). \end{aligned}$$

Therefore, $[f, f] \geq 0$, and thus $[\cdot, \cdot]$ is an inner product on $L^1(G)$ and hence satisfies the Cauchy-Schwarz inequality:

$$|[f, g]|^2 \leq [f, f][g, g], \quad f, g \in L^1(G).$$

Suppose that V is a symmetric neighborhood of 0 in G and define $g = \frac{\chi_V}{m(V)}$.

$$\begin{aligned} [f, g] - T_\phi(f) &= \int_G \int_G f(x)\overline{g(y)}\phi(x-y)dm(x)dm(y) - \int_G f(x)\phi(x)dm(x) \\ &= \int_G \frac{1}{m(V)} \int_V f(x)\phi(x-y)dm(y)dm(x) \\ &\quad - \int_G \frac{1}{m(V)} \int_V f(x)\phi(x)dm(y)dm(x) \\ &= \int_G f(x) \frac{1}{m(V)} \int_V (\phi(x-y) - \phi(x))dm(y)dm(x) \end{aligned}$$

and

$$[g, g] - 1 = \frac{1}{m(V)^2} \int_V \int_V (\phi(x-y) - 1)dm(x)dm(y).$$

Because ϕ is uniformly continuous, for any $\delta > 0$ there is some V such that both these integrals have absolute value $< \delta$, and then using the Cauchy-Schwarz inequality we get

$$|T_\phi(f)|^2 \leq [f, f], \quad f \in L^1(G). \quad (4)$$

Let $f \in L^1(G)$ and define $h = f * f^*$ and $h^n = h^{n-1} * h$ for $n \geq 2$. $|\phi(x)| \leq \phi(0) = 1$ tells us $\|\phi\|_\infty = 1$ and so $\|T_\phi\| \leq 1$. In fact, one checks that $\|T_\phi\| = 1$. Applying (4) to h, h^2, h^4, \dots we obtain

$$|T_\phi(f)|^2 \leq T_\phi(f * f^*) = T_\phi(h) \leq (T_\phi(h * h^*))^{1/2} = (T_\phi(h^2))^{1/2} \leq \dots,$$

so for any $n \geq 1$, because $\|T_\phi\| = 1$,

$$|T_\phi(f)|^2 \leq (T_\phi(h^{2^n}))^{2^{-n}} \leq \|h^{2^n}\|_1^{2^{-n}}.$$

The spectral radius formula tells us that

$$\lim_{n \rightarrow \infty} \|h^{2^n}\|_1^{2^{-n}} = \|\hat{h}\|_\infty.$$

But $\hat{h} = \widehat{ff^*} = |\hat{f}|^2$, so $|T_\phi(f)|^2 \leq \|\hat{f}\|_\infty^2$, i.e.

$$|T_\phi(f)| \leq \|\hat{f}\|_\infty, \quad f \in L^1(G).$$

We define S_ϕ on $A(\Gamma)$ by $S_\phi(\hat{f}) = T_\phi(f)$; this makes sense because if $\hat{f}_1 = \hat{f}_2$ then $|T_\phi(f_1 - f_2)| \leq \|\hat{f}_1 - \hat{f}_2\|_\infty = 0$, so $T_\phi(f_1) = T_\phi(f_2)$. The above inequality means that

$$|S_\phi(\hat{f})| \leq \|\hat{f}\|_\infty, \quad \hat{f} \in A(\Gamma).$$

Therefore S_ϕ is a bounded linear functional on $A(\Gamma)$, and because $A(\Gamma)$ is dense in $C_0(\Gamma)$, S_ϕ can be extended to a bounded linear functional on $C_0(\Gamma)$ with norm $\|S_\phi\| = 1$. But Γ is a locally compact Hausdorff space, so by the Riesz representation theorem there is a unique measure $\mu \in M(\Gamma)$ such that

$$S_\phi(g) = \int_\Gamma g(-\gamma) d\mu(\gamma), \quad g \in C_0(\Gamma),$$

and $\|\mu\| = \|S_\phi\| = 1$; we state the above with $g(-\gamma)$ rather than $g(\gamma)$ for later convenience. For $f \in L^1(G)$,

$$\begin{aligned} T_\phi(f) &= S_\phi(\hat{f}) \\ &= \int_\Gamma \hat{f}(-\gamma) d\mu(\gamma) \\ &= \int_\Gamma \int_G f(x) \langle -x, -\gamma \rangle dm(x) d\mu(\gamma) \\ &= \int_G f(x) \left(\int_\Gamma \langle x, \gamma \rangle d\mu(\gamma) \right) dm(x). \end{aligned}$$

But the definition of T_ϕ states

$$T_\phi(f) = \int_G f(x) \phi(x) dm(x).$$

Since these two expressions for $T_\phi(f)$ are equal for all $f \in L^1(G)$, we get that

$$\int_\Gamma \langle x, \gamma \rangle d\mu(\gamma) = \phi(x)$$

for almost all $x \in G$. Since both sides of the above equality are continuous, they are equal for all $x \in G$. For $x = 0$,

$$1 = \phi(0) = \int_\Gamma \langle 0, \gamma \rangle d\mu(\gamma) = \int_\Gamma d\mu(\gamma) \leq \int_\Gamma d|\mu|(\gamma) \leq \|\mu\| = 1.$$

Hence $\int_\Gamma d\mu(\gamma) = \int_\Gamma d|\mu|(\gamma)$, from which it follows that $\mu = |\mu|$, and therefore μ is a nonnegative measure. \square

6 The inversion theorem

Define $B(G)$ to be the set of those $f : G \rightarrow \mathbb{C}$ for which there is some $\mu_f \in M(\Gamma)$ such that

$$f(x) = \int_{\Gamma} \langle x, \gamma \rangle d\mu_f(\gamma), \quad x \in G.$$

It is apparent from Theorem 7 that there is at most one $\mu_f \in M(\Gamma)$ such that the above holds.

The following proof is from Rudin.¹⁴

Theorem 12 (Inversion theorem). *If $f \in L^1(G) \cap B(G)$, then $\hat{f} \in L^1(\Gamma)$.*

If the Haar measure m on G is fixed, then there is a Haar measure m_{Γ} on Γ such that for all $f \in L^1(G) \cap B(G)$,

$$f(x) = \int_{\Gamma} \hat{f}(\gamma) \langle x, \gamma \rangle dm_{\Gamma}(\gamma), \quad x \in G.$$

Proof. Write $B^1 = L^1(G) \cap B(G)$. For $f \in B^1$ and $h \in L^1(G)$,

$$\begin{aligned} (h * f)(0) &= \int_G h(x) f(-x) dm(x) \\ &= \int_G h(x) \int_{\Gamma} \langle -x, \gamma \rangle d\mu_f(\gamma) dm(x) \\ &= \int_{\Gamma} \hat{h}(\gamma) d\mu_f(\gamma). \end{aligned}$$

For $g \in B^1$, we have $h * g \in L^1(G)$ and $h * f \in L^1(G)$, and so using the above equality,

$$\int_{\Gamma} \widehat{h * g} d\mu_f = ((h * g) * f)(0) = ((h * f) * g)(0) = \int_{\Gamma} \widehat{h * f} d\mu_g,$$

hence

$$\int_{\Gamma} \hat{h} \hat{g} d\mu_f = \int_{\Gamma} \hat{h} \hat{f} d\mu_g, \quad f, g \in B^1, h \in L^1(G).$$

Because $A(\Gamma)$ is dense in $C_0(\Gamma)$ and the above holds for all $h \in L^1(G)$, it follows that

$$\hat{g} d\mu_f = \hat{f} d\mu_g, \quad f, g \in B^1. \quad (5)$$

We define $T : C_c(\Gamma) \rightarrow \mathbb{C}$ as follows. Let $\psi \in C_c(\Gamma)$, $K = \text{supp } \psi$. For $\gamma_0 \in K$, there is some $\phi \in C_0(\Gamma)$ such that $\phi(\gamma_0) \neq 0$, and because $A(\Gamma)$ is dense in $C_0(\Gamma)$ there is therefore some $f \in L^1(G)$ such that $\hat{f}(\gamma_0) \neq 0$. With $\delta = |\hat{f}(\gamma_0)|$, there is some $u \in C_c(G)$ with $\|u - f\|_1 < \delta$, and

$$|\hat{u}(\gamma_0) - \hat{f}(\gamma_0)| \leq \|u - f\|_1 < \delta,$$

¹⁴Walter Rudin, *Fourier Analysis on Groups*, p. 22, §1.5.1.

which shows that $\hat{u}(\gamma_0) \neq 0$. $\widehat{u * u^*} = |\hat{u}|^2 \geq 0$, so there is some open neighborhood U of γ_0 on which $\widehat{u * u^*}$ is positive, as it is a continuous function. Since K is compact, it is covered by finitely many of these open neighborhoods. Call the corresponding functions $u_1, \dots, u_n \in C_c(G)$, and write

$$g = u_1 * u_1^* + \dots + u_n * u_n^*.$$

g satisfies $\hat{g}(\gamma) > 0$ for all $\gamma \in K$. Because each u_n belongs to $C_c(G)$, $u_n * u_n^*$ belongs to $C_c(G)$ and so $g \in C_c(G)$. Moreover, by Theorem 9, each $u_n * u_n^*$ is positive-definite, and one checks that as g is a linear combination of positive-definite functions with nonnegative coefficients it is itself positive-definite. Because g is positive-definite, by Bochner's theorem it belongs to $B(G)$, and because $g \in C_c(G)$, g belongs to $L^1(G)$. Hence $g \in B^1$. We have now proved that there is at least one element of B^1 whose Fourier transform does not vanish on K . Suppose that f is any such function. Then using (5),

$$\int_{\Gamma} \frac{\psi}{\hat{f}} d\mu_f = \int_{\Gamma} \frac{\psi}{\hat{f}\hat{g}} \hat{g} d\mu_f = \int_{\Gamma} \frac{\psi}{\hat{f}\hat{g}} \hat{f} d\mu_g = \int_G \frac{\psi}{\hat{g}} d\mu_g.$$

Thus, it makes sense to define

$$T\psi = \int_{\Gamma} \frac{\psi}{\hat{g}} d\mu_g. \quad (6)$$

One checks that T is linear. Because g is positive-definite, the measure μ_g supplied by Bochner's theorem is nonnegative, and hence if $\psi \geq 0$ then $T\psi \geq 0$, namely, T is positive. There are $f \in B^1$ and $\psi \in C_c(G)$ such that $\int_{\Gamma} \psi d\mu_f \neq 0$, and $\psi\hat{f} \in C_c(G)$, so there is some $g \in B^1$ satisfying

$$T(\psi\hat{f}) = \int_{\Gamma} \frac{\psi\hat{f}}{\hat{g}} d\mu_g = \int_{\Gamma} \psi d\mu_f \neq 0,$$

showing that $T \neq 0$.

Let $\psi \in C_c(\Gamma)$ and $\gamma_0 \in \Gamma$. There is some $g \in B^1$ such that \hat{g} is positive on both K and $K + \gamma_0$. For $f(x) = \langle -x, \gamma_0 \rangle g(x)$, $x \in G$, we have $\hat{f}(\gamma) = \hat{g}(\gamma + \gamma_0)$, $\gamma \in \Gamma$, and $\mu_f(E) = \mu_g(E - \gamma_0)$. For $\psi_0 \in C_c(\Gamma)$ defined by $\psi_0(\gamma) = \psi(\gamma - \gamma_0)$,

$$T\psi_0 = \int_{\Gamma} \frac{\psi(\gamma - \gamma_0)}{\hat{g}(\gamma)} d\mu_g(\gamma) = \int_{\Gamma} \frac{\psi(\gamma)}{\hat{f}(\gamma)} d\mu_f(\gamma) = T\psi,$$

showing that T is translation invariant. Then by the Riesz representation theorem, there is some nonnegative regular measure m_{Γ} on Γ satisfying

$$T\psi = \int_{\Gamma} \psi dm_{\Gamma}, \quad \psi \in C_c(\Gamma).$$

This measure m_{Γ} is translation invariant and not the zero measure because T has these properties, and this means that it is a Haar measure on Γ .

For $f \in B^1$,

$$\int_{\Gamma} \psi d\mu_f = T(\psi \hat{f}) = \int_{\Gamma} \psi \hat{f} dm_{\Gamma}, \quad \psi \in C_c(\Gamma),$$

which implies that

$$d\mu_f = \hat{f} dm_{\Gamma}, \quad f \in B^1.$$

Because $\|\mu_f\| < \infty$ (as $\mu_f \in M(\Gamma)$), the above equality implies that $\hat{f} \in L^1(\Gamma)$. Moreover, by the definition of μ_f , for any $x \in G$ we have

$$f(x) = \int_{\Gamma} \langle x, \gamma \rangle d\mu_f(\gamma) = \int_{\Gamma} \langle x, \gamma \rangle \hat{f} dm_{\Gamma}(\gamma).$$

□

Using the inversion theorem, we prove the following lemma.¹⁵

Lemma 13. $\{x_0 + V(C, r) : x_0 \in G, C \text{ is a compact subset of } G, r > 0\}$ is a basis for the topology of G .

Γ separates points in G .

Proof. Let m_{Γ} be the Haar measure on Γ specified in the inversion theorem. Suppose that V is a neighborhood of 0 in G . Let W be a compact neighborhood of 0 in G satisfying $W - W \subset V$. (One proves that there are such W .) Define $f = \frac{\chi_W}{\sqrt{m(W)}}$ and $g = f * f^*$. g is continuous and positive-definite, and $\text{supp } g \subset W - W$. Because g is continuous and positive-definite, by Bochner's theorem it belongs to $B(G)$, and because $\text{supp } g \subset W - W$ it belongs to $L^1(G)$, so we can apply the inversion theorem to get $\hat{g} \in L^1(\Gamma)$ and

$$\int_{\Gamma} \hat{g} dm_{\Gamma}(\gamma) = g(0) = \int_G f(-y) \overline{f(-y)} dm(y) = \int_G |f(y)|^2 dm(y) = 1.$$

Because $\hat{g} = |\hat{f}|^2 \geq 0$, there is a compact set C in Γ such that

$$\int_C \hat{g}(\gamma) dm_{\Gamma}(\gamma) > \frac{2}{3}.$$

To say that $x \in V(C, 1/3)$ means $|1 - \langle x, \gamma \rangle| < \frac{1}{3}$ for all $\gamma \in C$ and hence $\text{Re } \langle x, \gamma \rangle > \frac{2}{3}$, and satisfies

$$g(x) = \text{Re} \int_C \hat{g}(\gamma) \langle x, \gamma \rangle dm_{\Gamma}(\gamma) + \text{Re} \int_{C'} \hat{g}(\gamma) \langle x, \gamma \rangle dm_{\Gamma}(\gamma).$$

The first term is $> \frac{4}{9}$ and the second term has absolute value $< \frac{1}{3}$, so $g(x) > \frac{1}{9}$ for $x \in V(C, 1/3)$. But $g(x) > \frac{1}{9}$ means that $x \in \text{supp } g = W - W \subset V$, so

$$V(C, 1/3) \subset V,$$

¹⁵Walter Rudin, *Fourier Analysis on Groups*, p. 23, §1.5.2.

from which it follows that $V(C, r)$, C compact and $r > 0$, is a local basis at 0.

For $x_0 \in G$, $x_0 \neq 0$, let V be a neighborhood of 0 that does not include x_0 , and the above gives $x \notin V(C, 1/3)$, i.e., there is some $\gamma \in \Gamma$ such that $|1 - \langle x, \gamma \rangle| \geq \frac{1}{3}$, and hence $\langle x, \gamma \rangle \neq 1$. Therefore, if $x_1 \neq x_2$, there is some $\gamma \in \Gamma$ such that $\langle x_1 - x_2, \gamma \rangle \neq 1$, and hence $\langle x_1, \gamma \rangle \neq \langle x_2, \gamma \rangle$, which is what it means to say that Γ separates points in G . \square

7 Pontryagin duality theorem

The dual group Γ of G is itself a locally compact abelian group and so has a dual group $\widehat{\Gamma}$. We proved in Theorem 3 that $(x, \gamma) \mapsto \langle x, \gamma \rangle$ is continuous, and therefore for any $x \in G$, the function $\alpha(x) : \Gamma \rightarrow \mathbb{C}$ defined by

$$\langle \gamma, \alpha(x) \rangle = \langle x, \gamma \rangle, \quad \gamma \in \Gamma,$$

belongs to $\widehat{\Gamma}$. For $x, y \in G$, $\alpha(xy) \in \widehat{\Gamma}$ satisfies

$$\langle \gamma, \alpha(xy) \rangle = \langle xy, \gamma \rangle = \langle x, \gamma \rangle \langle y, \gamma \rangle = \langle \gamma, \alpha(x) \rangle \langle \gamma, \alpha(y) \rangle,$$

showing that $\alpha : G \rightarrow \widehat{\Gamma}$ is a homomorphism. The following theorem, proved in Rudin,¹⁶ shows that α is an isomorphism of topological groups. That is, it states that a locally compact abelian group is isomorphic as a topological group to its double dual. Let **LCA** denote the category of locally compact abelian groups, where morphisms are continuous group homomorphisms. Taking the double dual of an element of **LCA** is a functor, and it can be proved that there is a natural isomorphism between the identity functor in **LCA** and the double dual functor.¹⁷

Theorem 14 (Pontryagin duality theorem). $\alpha : G \rightarrow \widehat{\Gamma}$ defined by

$$\langle \gamma, \alpha(x) \rangle = \langle x, \gamma \rangle, \quad \gamma \in \Gamma,$$

is an isomorphism of topological groups.

We proved earlier that if G is discrete then Γ is compact and that if G is compact then Γ is discrete, but had not established that if Γ is compact then G is discrete or if Γ is discrete then G is compact, but we obtain these conclusions from the Pontryagin duality theorem: if Γ is compact then $\widehat{\Gamma}$ is discrete, and G is isomorphic as a topological group to $\widehat{\Gamma}$ so G is discrete, and likewise if Γ is discrete then G is compact.

8 Further reading

Keith Conrad, <http://www.math.uconn.edu/~kconrad/blurbs/gradnumthy/characterQ.pdf> works out explicitly the form of all characters of \mathbb{Q} , and shows

¹⁶Walter Rudin, *Fourier Analysis on Groups*, p. 28, §1.7.2.

¹⁷For more, see the nLab page: <http://ncatlab.org/nlab/show/Pontryagin+dual>

that the group of all characters of \mathbb{Q} (namely, the dual group of \mathbb{Q} when \mathbb{Q} has the discrete topology), is isomorphic as a group to the quotient group $\hat{\mathbb{A}}_{\mathbb{Q}}/\mathbb{Q}$. This gives a satisfying reason for caring about the p -adic numbers \mathbb{Q}_p and the adèles $\hat{\mathbb{A}}_{\mathbb{Q}}$.