L^p norms of trigonometric polynomials

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A *trigonometric polynomial of degree* n is an expression of the form

$$\sum_{k=-n}^{n} c_k e^{ikt}, \qquad c_k \in \mathbb{C}.$$

Using the identity $e^{it} = \cos t + i \sin t$, we can write a trigonometric polynomial of degree n in the form

$$a_0 + \sum_{k=1}^{n} a_k \cos kt + \sum_{k=1}^{n} b_k \sin kt, \qquad a_k, b_k \in \mathbb{C}.$$

For $1 \leq p < \infty$ and for a 2π-periodic function f, we define the L^p norm of f by

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^p dt \right)^{1/p}.$$

For a continuous 2π-periodic function f, we define the L^∞ norm of f by

$$\|f\|_\infty = \max_{0 \leq t \leq 2\pi} |f(t)|.$$

If f is a continuous 2π-periodic function, then there is a sequence of trigonometric polynomials f_n such that $\|f - f_n\|_\infty \to 0$ as $n \to \infty$ [31, p. 54, Corollary 5.4].

If $1 \leq p < \infty$ and f is a continuous 2π-periodic function, then

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^p dt \right)^{1/p} \leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|f\|_\infty^p dt \right)^{1/p} = \|f\|_\infty.$$

Jensen's inequality [16, p. 44, Theorem 2.2] (cf. [30, p. 113, Problem 7.5]) tells us that if $\phi : [0, \infty) \to \mathbb{R}$ is convex, then for any function $h : [0, 2\pi] \to [0, \infty)$ we have

$$\phi\left(\frac{1}{2\pi} \int_0^{2\pi} h(t) dt \right) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(h(t)) dt.$$

If $1 \leq p < q < \infty$, then $\phi : [0, \infty) \to \mathbb{R}$ defined by $\phi(x) = x^{q/p}$ is convex. Hence, if $1 \leq p < q < \infty$ then for any 2π-periodic function f,

$$\begin{aligned}
\|f\|_p &= (\phi(\|f\|_p^p))^{1/q} \\
&= \left(\phi\left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p dt\right)\right)^{1/q} \\
&\leq \left(\frac{1}{2\pi}\int_0^{2\pi} \phi(|f(t)|^p) dt\right)^{1/q} \\
&= \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^q dt\right)^{1/q} \\
&= \|f\|_q.
\end{aligned}$$

The *Dirichlet kernel* D_n is defined by

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt} = 1 + 2\sum_{k=1}^{n} \cos kt.$$

One can show [14, p. 71, Exercise 1.1] that

$$\|D_n\|_1 = \frac{4}{\pi^2} \cdot \log n + O(1).$$

(On the other hand, it can quickly be seen that $\|D_n\|_\infty = 2n + 1$, and it follows from Parseval's identity that $\|D_n\|_2 = \sqrt{2n+1}$.)

Pólya and Szegő [27, Part VI] present various problems about trigonometric polynomials together with solutions to them. A result on L^∞ norms of trigonometric polynomials that Pólya and Szegő present is for the sum $A_n(t) = \sum_{k=1}^{n} \frac{\sin kt}{k}$. The local maxima and local minima of A_n can be explicitly determined [27, p. 74, no. 23], and it can be shown that [27, p. 74, no. 25]

$$\|A_n\|_\infty \sim \int_0^\pi \frac{\sin t}{t} dt.$$

1 L^p norms

If $1 \leq p < q < \infty$, then [14, p. 123, Exercise 1.8] (cf. [7, p. 102, Theorem 2.6]) there is some $C(p, q)$ such that for any trigonometric polynomial f of degree n, we have

$$\|f\|_q \leq C(p, q) n^{\frac{1}{p} - \frac{1}{q}} \|f\|_p.$$

This inequality is sharp [33, p. 230]: for $1 \leq p < q < \infty$ there is some $C'(p, q)$ such that if $F_n(t) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(t)$ (F_n is called the *Fejér kernel*) then

$$\|F_n\|_q > C'(p, q) n^{\frac{1}{p} - \frac{1}{q}} \|F_n\|_p.$$

Let $X_n = \{a_0 + \sum_{k=1}^n a_k \cos kt + b_k \sin kt : a_k, b_k \in \mathbb{R}\}$, the real vector space of real valued trigonometric polynomials of degree n, have norm

$$\|f\|_{X_n} = \max\{|a_0|, |a_1|, \dots, |a_n|, |b_1|, \dots, |b_n|\}.$$

Let $Y_{n,p}$ be the same vector space with the L^p norm. Ash and Ganzburg [1] give upper and lower bounds on the operator norm of the map $i : X_n \to Y_{n,p}$ defined by $i(f) = f$.

Bernstein's inequality [14, p. 50, Exercise 7.16] states that for $1 \leq p \leq \infty$, if f is a trigonometric polynomial of degree n, then

$$\|f'\|_p \leq n\|f\|_p.$$

In the other direction, if $f \in C^1$ then

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} f(s)ds + \frac{1}{2\pi}\int_0^t sf'(s)ds + \frac{1}{2\pi}\int_t^{2\pi}(s-2\pi)f'(s)ds \\
=&\frac{1}{2\pi}\int_0^{2\pi} f(s)ds + \frac{1}{2\pi}\int_0^{2\pi} sf'(s)ds - \int_t^{2\pi} f'(s)ds \\
=&\frac{1}{2\pi}\int_0^{2\pi} f(s)ds + \frac{1}{2\pi}sf(s)\Big|_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} f(s)ds - f(s)\Big|_t^{2\pi} \\
=&f(t).
\end{aligned}$$

Hence

$$\begin{aligned}
|f(t)| &\leq \frac{1}{2\pi}\int_0^{2\pi}|f(s)|ds + \frac{1}{2\pi}\int_0^t s|f'(s)|ds + \frac{1}{2\pi}\int_t^{2\pi}(2\pi - s)|f'(s)|ds \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}|f(s)|ds + \int_0^t |f'(s)|ds + \int_t^{2\pi}|f'(s)|ds \\
&= \|f\|_1 + 2\pi\|f'\|_1,
\end{aligned}$$

so

$$\|f\|_\infty \leq \|f\|_1 + 2\pi\|f'\|_1.$$

This is an instance of the Sobolev inequality [26].

It turns out that for a trigonometric polynomial the mass cannot be too concentrated. More precisely, the number of nonzero terms of a trigonometric polynomial restricts how concentrated its mass can be. Let $d\mu = \frac{dt}{2\pi}$. Thus $\mu([0,2\pi]) = 1$. A result of Turán [20, p. 89, Lemma 1] states that if $\lambda_1, \dots, \lambda_N \in \mathbb{Z}$ and $T(t) = \sum_{n=1}^N b_n e^{i\lambda_n t}$, $b_n \in \mathbb{C}$, then for any closed arc $I \subset [0, 2\pi]$,

$$\|T\|_\infty \leq \left(\frac{2e}{\mu(I)}\right)^{N-1} \max_{t\in I}|T(t)|.$$

Nazarov [11, p. 452] shows that there is some constant A such that if E is a closed subset of $[0, 2\pi]$ (not necessarily an arc), then

$$\|\hat{T}\|_1 \leq \left(\frac{A}{\mu(E)}\right)^N \max_{t\in E}|f(T)|.$$

Nazarov [23] proves that there exists some constant C such that if $0 \le q \le 2$ and $\mu(E) \ge \frac{1}{3}$, then

$$\|T\|_q \le e^{C(N-1)\left(1-\frac{\mu(E)}{2\pi}\right)} \left(\frac{1}{2\pi}\int_E |T(t)|^q dt\right)^{1/q}.$$

These results of Turan and Nazarov are examples of the *uncertainty principle* [9], which is the general principle that a constrain on the support of the Fourier transform of a function constrains the support of the function itself.

In [10], Hardy and Littlewood present inequalities for norms of 2π-periodic functions in terms of certain series formed from their Fourier coefficients. Let $c_k \in \mathbb{C}$, $k \in \mathbb{Z}$, be such that $c_k \to 0$ as $k \to \pm\infty$, and define $c_0^*, c_1^*, c_{-1}^*, c_2^*, c_{-2}^*, \ldots$ to be the absolute values of the c_k ordered in decreasing magnitude. For real $r > 1$, define

$$S_r^*(c) = \left(\sum_{k=-\infty}^{\infty} c_k^{*\,r}(|k|+1)^{r-2}\right)^{1/r}.$$

For instance, if $c_k = 1$ for $-N \le k \le N$ and $c_k = 0$ for $|k| > N$, then $S_r^*(c) = \left(1 + 2\sum_{k=2}^{N+1} k^{r-2}\right)^{1/r}$. Hardy and Littlewood state the result [10, p. 164, Theorem 2] that if $1 < p \le 2$ then there is some constant $A(p)$ such that for any sequence c, with $c_k \to 0$ as $k \to \pm\infty$, if $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$ and $\|f\|_p < \infty$ then

$$S_p^*(c) \le A(p)\|f\|_p.$$

A proof of this is given in Zygmund [35, vol. II, p. 128, chap. XII, Theorem 6.3]. Asking if this inequality holds for $p = 1$ suggests the following question that Hardy and Littlewood pose at the end of their paper [10, p. 168]: Is there a constant A such that for all distinct positive integers $m_k, k = 1, \ldots, N$, we have

$$\|\sum_{k=1}^{N} \cos m_k t\|_1 > A \log N?$$

McGehee, Pigno and Smith [18] prove that there is some K such that for all N, if $n_1, \ldots, n_N$ are distinct integers and $c_1, \ldots, c_N \in \mathbb{C}$ satisfy $|c_k| \ge 1$, then

$$\|\sum_{k=1}^{N} c_k e^{in_k t}\|_1 > K \log N.$$

Thus

$$\|\sum_{k=1}^{N} \cos m_k t\|_1 = \frac{1}{2} \cdot \|\sum_{k=1}^{N} e^{im_k t} + e^{-im_k t}\|_1 \ge \frac{1}{2} \cdot K \log(2N).$$

For $k \ge 2$, define $T_N(t) = \sum_{n=1}^{N} e^{in^k t}$. Since $\|T_N\|_\infty = N$, for each $p \ge 1$ we have $\|T_N\|_p \le N$. Hua's lemma [22, p. 116, Theorem 4.6] states that if $\epsilon > 0$, then

$$\|T_N\|_{2^k} = O\left(N^{1-\frac{k}{2^k}+\epsilon}\right).$$

Hua's lemma is used in additive number theory. The number of sets of integer solutions of the equation

$$f(x_1,\dots,x_n)=N, \qquad a_r \le x_r \le b_r$$

is equal to (cf. [12, p. 151])

$$\sum_{a_1\le x_1\le b_1}\cdots\sum_{a_n\le x_n\le b_n}\int_0^1 e^{2\pi i(f(x_1,\dots,x_n)-N)t}dt.$$

Borwein and Lockhart [4]: what is the expected L^p norm of a trigonometric polynomial of order n? Kahane [13, Chapter 6] also presents material on random trigonometric polynomials.

Nursultanov and Tikhonov [25]: the sup on a subset of $\mathbb{T}$ of a trigonometric polynomial f of degree n being lower bounded in terms of $\|f\|_\infty$, n, and the measure of the subset.

2 ℓ^p norms

For a 2π-periodic function f, we define $\hat{f}:\mathbb{Z}\to\mathbb{C}$ by

$$\hat{f}(k)=\frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}f(t)dt.$$

For $1\le p<\infty$, we define the ℓ^p norm of $\hat{f}$ by

$$\|\hat{f}\|_p=\left(\sum_{k=-\infty}^{\infty}|\hat{f}(k)|^p\right)^{1/p},$$

and we define the ℓ^∞ norm of $\hat{f}$ by

$$\|\hat{f}\|_\infty=\max_{k\in\mathbb{Z}}|\hat{f}(k)|.$$

Parseval's identity [31, p. 80, Theorem 1.3] states that $\|f\|_2=\|\hat{f}\|_2$.

If $1\le p<\infty$, then

$$\|\hat{f}\|_\infty \le \left(\cdots+\|\hat{f}\|_\infty^p+\cdots\right)^{1/p}=\|\hat{f}\|_p.$$

If $1\le p<q<\infty$, then, since for each k, $\frac{|\hat{f}(k)|}{\|\hat{f}\|_q}\le 1$,

$$1=\left(\sum_{k=-\infty}^{\infty}\left(\frac{|\hat{f}(k)|}{\|\hat{f}\|_q}\right)^q\right)^{1/q}\le\left(\sum_{k=-\infty}^{\infty}\left(\frac{|\hat{f}(k)|}{\|\hat{f}\|_q}\right)^p\right)^{1/q}=\frac{\|\hat{f}\|_p^{p/q}}{\|\hat{f}\|_q^{p/q}}.$$

Hence for $1\le p<p\le\infty$,

$$\|\hat{f}\|_q\le\|\hat{f}\|_p.$$

For $1 \le p < \infty$, if f is a trigonometric polynomial of degree n then

$$\|\hat{f}\|_p = \left(\sum_{k=-n}^{n} |\hat{f}(k)|^p \right)^{1/p} \le \left(\sum_{k=-n}^{n} \|\hat{f}\|_\infty^p \right)^{1/p} = (2n+1)^{1/p} \|\hat{f}\|_\infty.$$

For $1 \le p < q < \infty$, we have [30, p. 123, Problem 8.3] (this is Jensen's inequality for sums)

$$\left(\sum_{k=-n}^{n} \frac{1}{2n+1} |\hat{f}(k)|^p \right)^{1/p} \le \left(\sum_{k=-n}^{n} \frac{1}{2n+1} |\hat{f}(k)|^q \right)^{1/q},$$

i.e.

$$(2n+1)^{-1/p} \|\hat{f}\|_p \le (2n+1)^{-1/q} \|\hat{f}\|_q.$$

Hence for $1 < p < q < \infty$,

$$\|\hat{f}\|_p \le (2n+1)^{\frac{1}{p}-\frac{1}{q}} \|\hat{f}\|_q.$$

For any t,

$$|f(t)| = \left| \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikt} \right| \le \sum_{k=-\infty}^{\infty} |\hat{f}(k) e^{ikt}| = \sum_{k=-\infty}^{\infty} |\hat{f}(k)| = \|\hat{f}\|_1.$$

Hence

$$\|f\|_\infty \le \|\hat{f}\|_1.$$

For any $k \in \mathbb{Z}$,

$$|\hat{f}(k)| = \left| \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt \right| \le \frac{1}{2\pi} \int_0^{2\pi} |f(t)| dt = \|f\|_1.$$

Hence

$$\|\hat{f}\|_\infty \le \|f\|_1.$$

The Hausdorff-Young inequality [32, p. 57, Corollary 2.4] states that for $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, if $f \in L^p$ then

$$\|\hat{f}\|_q \le \|f\|_p.$$

The dual Hausdorff-Young inequality [32, p. 58, Corollary 2.5] states that for $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, if $f \in L^q$ then

$$\|f\|_q \le \|\hat{f}\|_q.$$

A survey on the Hausdorff-Young inequality is given in [6])

For $M + 1 \le k \le M + N$, let $a_k \in \mathbb{C}$ and let $S(t) = \sum_{k=M+1}^{N+1} a_k e^{ikt}$. Let $t_1, \ldots, t_R \in \mathbb{R}$, and let δ be such that if $r \neq s$ then

$$\|t_r - t_s\| \ge \delta,$$

where $\|t\| = \min_k |t-k|$ is the distance from t to a nearest integer. *The large sieve* [19] is an inequality of the form

$$\sum_{r=1}^{R} |S(2\pi t_r)|^2 \le \Delta(N,\delta) \sum_{k=M+1}^{M+N} |a_k|^2.$$

A result of Selberg [19, p. 559, Theorem 3] shows that the large sieve is valid for $\Delta = N - 1 + \delta^{-1}$.

Kristiansen [15]

Boas [2]

For $F : \mathbb{Z}/n \to \mathbb{C}$, its Fourier transform $\hat{F} : \mathbb{Z}/n \to \mathbb{C}$ (called the *discrete Fourier transform*) is defined by

$$\hat{F}(k) = \frac{1}{n} \sum_{j=0}^{n-1} F(j) e^{-2\pi i jk/n}, \qquad 0 \le k \le n-1,$$

and one can prove [31, p. 223, Theorem 1.2] that

$$F(j) = \sum_{k=0}^{n-1} \hat{F}(k) e^{2\pi i kj/N}, \qquad 0 \le j \le n-1.$$

One can also prove Parseval's identity for the Fourier transform on $\mathbb{Z}/n$ [31, p. 223, Theorem 1.2]. It states

$$\sum_{k=0}^{n-1} |\hat{F}(k)|^2 = \frac{1}{n} \sum_{j=0}^{n-1} |F(j)|^2.$$

Let $P(t) = \sum_{k=0}^{n-1} a_k e^{ikt}$. Define $F : \mathbb{Z}/n \to \mathbb{C}$ by

$$F(j) = \sum_{k=0}^{n-1} a_k e^{2\pi i kj/n}, \qquad 0 \le j \le n-1.$$

(That is, $\hat{F}(k) = a_k$.) We then have

$$\sum_{k=0}^{n-1} |a_k|^2 = \frac{1}{n} \sum_{j=0}^{n-1} |F(j)|^2 = \frac{1}{n} \sum_{j=0}^{n-1} \left|P\left(\frac{2\pi j}{n}\right)\right|^2.$$

Thus

$$\|P\|_2 = \left(\frac{1}{n} \sum_{j=0}^{n-1} \left|P\left(\frac{2\pi j}{n}\right)\right|^2 \right)^{1/2}.$$

The Marcinkiewicz-Zygmund inequalities [35, vol. II, p. 28, chap. X, Theorem 7.5] state that there is a constant A such that for $1 \le p \le \infty$, if f is a trigonometric polynomial of degree n then

$$\left(\frac{1}{2n+1} \sum_{k=0}^{2n} \left|f\left(\frac{2\pi k}{2n+1}\right)\right|^p \right)^{1/p} \le A(2\pi)^{1/p} \|f\|_p,$$

and for each $1 < p < \infty$ there exists some A_p such that if f is a trigonometric polynomial of degree n then

$$\|f\|_p \le A_p \left(\frac{1}{2n+1} \sum_{k=0}^{2n} \left| f\Big(\frac{2\pi k}{2n+1}\Big)\right|^p \right)^{1/p}.$$

Máté and Nevai [17, p. 148, Theorem 6] prove that for $p > 0$, if S_n is a trigonometric polynomial of degree n then

$$\|S_n\|_\infty \le \left(\frac{(1+np)e}{2} \right)^{1/p} \|S_n\|_p.$$

Máté and Nevai [17] prove a version of Bernstein's inequality for $0 < p < 1$, and their result can be sharpened to the following [34]: For $0 < p < 1$, if T_n is a trigonometric polynomial of order n then

$$\|T_n'\|_p \le n\|T_n\|_p.$$

Let $\operatorname{supp} \hat{f} = \{k \in \mathbb{Z} : \hat{f}(k) \neq 0\}$. A subset Λ of $\mathbb{Z}$ is called a *Sidon set* [28, p. 121, §5.7.2] if there exists a constant B such that for every trigonometric polynomial f with $\operatorname{supp} \hat{f} \subseteq \Lambda$ we have

$$\|\hat{f}\|_1 \le B\|f\|_\infty.$$

Let $B(\Lambda)$ be the least such B. A sequence of positive integers λ_k is said to be *lacunary* if there is a constant ρ such that $\lambda_{k+1} > \rho\lambda_k$ for all k. If λ_k is a lacunary sequence, then $\{\lambda_k\}$ is a Sidon set [21, p. 154, Corollary 6.17]. If $\Lambda \subset \mathbb{Z}$ is a Sidon set, then [28, p. 128, Theorem 5.7.7] (cf. [21, p. 157, Corollary 6.19]) for any $2 < p < \infty$, for every trigonometric polynomial f with $\operatorname{supp} \hat{f} \subseteq \Lambda$ we have

$$\|f\|_p \le B(\Lambda)\sqrt{p}\|f\|_2,$$

and

$$\|f\|_2 \le 2B(\Lambda)\|f\|_1.$$

Let $0 < p < \infty$. A subset E of $\mathbb{Z}$ is called a *$\Lambda(p)$-set* if for every $0 < r < p$ there is some $A(E,p)$ such that for all trigonometric polynomials f with $\operatorname{supp} \hat{f} \subset E$ we have

$$\|f\|_p \le A(E,p)\|f\|_2.$$

$\Lambda(p)$ sets were introduced by Rudin, and he discusses them in his autobiography [29, Chapter 28]. A modern survey of $\Lambda(p)$-sets is given by Bourgain [5].

Bochkarev [3] proves various lower bounds on the L^1 norms of certain trigonometric polynomials. Let $c_k \in \mathbb{C}$, $k \ge 1$. If there are constants A and B such that

$$A\frac{(\log k)^s}{\sqrt{k}} \le |c_k| \le B\frac{(\log k)^s}{\sqrt{k}}, \qquad k \ge 1,$$

then [3, p. 58, Theorem 19]

$$\|\sum_{k=1}^{n} c_k e^{ik^2 t}\|_1 \gg \begin{cases} (\log n)^{s-\frac{1}{2}}, & s > \frac{1}{2}, \\ \log\log n, & s = \frac{1}{2}. \end{cases}$$

If $P(t) = \sum_{k=0}^{n} a_k e^{ikt}$ with $a_k \in \{-1, 1\}$, then by the Cauchy-Schwarz inequality and Parseval's identity we have

$$\|P\|_1 = \frac{1}{2\pi}\int_0^{2\pi} 1 \cdot |P(t)| dt \le \|1\|_2 \cdot \|P\|_2 = 1 \cdot \|\hat{P}\|_2 = \sqrt{n+1}.$$

Newman [24] shows that in fact we can do better than what we get using the Cauchy-Schwarz inequality and Parseval's identity:

$$\|P\|_1 < \sqrt{n + 0.97}.$$

A *Fekete polynomial* is a polynomial of the form $\sum_{k=1}^{l-1} \left(\frac{k}{l}\right) z^k$, l prime, where $\left(\frac{k}{l}\right)$ is the Legendre symbol. Let $P_l(t) = \sum_{k=1}^{l-1} \left(\frac{k}{l}\right) e^{ikt}$. Erdélyi [8] proves upper and lower bounds on $\left(\frac{1}{|I|}\int_I |P_l(t)|^q dt\right)^{1/q}$, $q > 0$, where I is an arc in $[0, 2\pi]$.

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