Convolution semigroups, canonical processes, and Brownian motion

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1 Convolution semigroups, projective families, and canonical processes

Let

$$E = \mathbb{R}^d$$

and let $\mathscr{E} = \mathscr{B}_{\mathbb{R}^d}$, the Borel σ-algebra of $\mathbb{R}^d$, and let $\mathscr{P}(E)$ be the collection of Borel probability measures on $\mathbb{R}^d$. With the **narrow topology**, $\mathscr{P}(E)$ is a Polish space.[1] For a nonempty set J, we write

$$\mathscr{E}^J = \bigotimes_{t \in J} \mathscr{E},$$

the product σ-algebra.

Let $A : E \times E \to E$ be $A(x_1, x_2) = x_1 + x_2$. For $\nu_1, \nu_2 \in \mathscr{P}(E)$, the **convolution of** ν_1 **and** ν_2 is the pushforward of the product measure $\nu_1 \times \nu_2$ by A:

$$\nu_1 * \nu_2 = A_*(\nu_1 \times \nu_2).$$

The convolution $\nu_1 * \nu_2$ is an element of $\mathscr{P}(E)$.

Let

$$I = \mathbb{R}_{\geq 0}.$$

A **convolution semigroup** is a family $(\nu_t)_{t \in I}$ of elements of $\mathscr{P}(E)$ such that for $s, t \in I$,

$$\nu_{s+t} = \nu_s * \nu_t.$$

From this, it turns out that $\mu_0 = \delta_0$.[2] A convolution semigroup is called **continuous** when the map $t \mapsto \nu_t$ is continuous $I \to \mathscr{P}(E)$.

[1] See http://individual.utoronto.ca/jordanbell/notes/narrow.pdf
[2] http://individual.utoronto.ca/jordanbell/notes/markovkernels.pdf, Theorem 3.

For $\nu \in \mathscr{P}(E)$ and $x \in E$, and for $B \in \mathscr{E}$,

$$(\nu * \delta_x)(B) = \int_E \left(\int_E 1_B(x_1 + x_2) d\delta_x(x_1) \right) d\nu(x_2) = \nu(B - x),$$

and we define $\nu^x \in \mathscr{P}(E)$ by

$$\nu^x = \nu * \delta_x.$$

For $\nu \in \mathscr{P}(E)$ and for a Borel measurable function $f : E \to [0, \infty]$, write

$$\nu f = \int_E f d\mu.$$

For $x \in E$, using the change of variables formula[3] and Fubini's theorem,

$$\begin{aligned} \nu^x f &= \int_E f d(\nu * \delta_x) \\ &= \int_{E \times E} f \circ A d(\nu \times \delta_x) \\ &= \int_E \left(\int_E f(x_1 + x_2) d\delta_x(x_2) \right) d\nu(x_1) \\ &= \int_E f(x_1 + x) d\nu(x_1). \end{aligned}$$

That is, for $\nu \in \mathscr{P}(E)$, for $f : E \to [0, \infty]$ Borel measurable, and for $x \in E$,

$$\nu^x f = \int_E f d\nu^x = \int_E f(x + y) d\nu(y). \tag{1}$$

For nonempty subsets J and K of I with $J \subset K$, let

$$\pi_{K,J} : E^K \to E^J$$

be the projection map. Let $\mathscr{K} = \mathscr{K}(I)$ be the collection of finite nonempty subsets of I. Let $(\Omega, \mathscr{F}, P, (X_t)_{t \in I})$ be a stochastic process with state space E. For $J \in \mathscr{K}$, with elements $t_1 < \ldots < t_n$, we define

$$X_J = X_{t_1} \otimes \cdots \otimes X_{t_n},$$

which is measurable $\mathscr{F} \to \mathscr{E}^J$. The **joint distribution** P_J of the family of random variables $(X_t)_{t \in J}$ is the distribution of X_J, i.e.

$$P_J = X_{J*} P.$$

The **family of finite-dimensional distributions** of X is the family $(P_J)_{J \in \mathscr{K}}$. For $J, K \in \mathscr{K}$ with $J \subset K$,

$$X_J = \pi_{K,J} \circ X_K,$$

[3] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 484, Theorem 13.46.

from which

$$(\pi_{K,J})_* P_K = P_J. \tag{2}$$

Forgetting the stochastic process X, a family of probability measures P_J on $\mathscr{E}^J$, for $J \in \mathscr{K}$, is called a **projective family** when (2) is true. The **Kolmogorov extension theorem**[4] tells us that if $(P_J)_{J\in\mathscr{K}}$ is a projective family, then there is a unique probability measure P_I on $\mathscr{E}^I$ such that for any $J \in \mathscr{K}$,

$$(\pi_{I,J})_* P_I = P_J. \tag{3}$$

Then for $\Omega = E^I$ and $\mathscr{F} = \mathscr{E}^I$, $(\Omega, \mathscr{F}, P_I)$ is a probability space, and for $t \in I$ we define $X_t : \Omega \to E$ by

$$X_t(\omega) = \pi_{I,\{t\}}(\omega) = \omega(t), \tag{4}$$

which is measurable $\mathscr{F} \to \mathscr{E}$, and thus the family $(X_t)_{t\in I}$ is a stochastic process with state space E. For $J \in \mathscr{K}$ it is immediate that

$$X_J = \pi_{I,J}.$$

For $B \in \mathscr{E}^J$, applying (3) gives

$$(X_{J*}P_I)(B) = ((\pi_{I,J})_* P_I)(B) = P_J(B),$$

which means that $X_{J*}P_I = P_J$, namely, $(P_J)_{J\in\mathscr{K}}$ is the family of finite-dimensional distributions of the stochastic process $(X_t)_{t\in I}$. We call the stochastic process (4) the **canonical process associated with the projective family** $(P_J)_{J\in\mathscr{K}}$.

Let $(\nu_t)_{t\in I}$ be a convolution semigroup and let $\mu \in \mathscr{P}(E)$. For $J \in \mathscr{K}$, with elements $t_1 < \ldots < t_n$, and for $B \in \mathscr{E}^J$, define

$$\begin{aligned} &P_J(B) \\ &= \int_E \int_E \cdots \int_E 1_B(x_1, \ldots, x_n) d\nu^{x_{n-1}}_{t_n - t_{n-1}}(x_n) \cdots d\nu^{x_0}_{t_1}(x_1) d\mu(x_0). \end{aligned} \tag{5}$$

We say that $(P_J)_{J\in\mathscr{K}}$ **is the family of measures induced by the convolution semigroup** $(\nu_t)_{t\in I}$. It is proved that $(P_J)_{J\in\mathscr{K}}$ is a projective family.[5] Therefore, from the Kolmogorov extension theorem it follows that there is a unique probability measure P^μ on $\mathscr{E}^I$ such that

$$(\pi_{I,J})_* P^\mu = P_J. \tag{6}$$

For $\Omega = E^I$ and $\mathscr{F} = \mathscr{E}^I$, $(\Omega, \mathscr{F}, P^\mu)$ is a probability space. For $t \in I$ define $X_t : \Omega \to E$ by

$$X_t(\omega) = \pi_{I,\{t\}}(\omega) = \omega(t).$$

[4] See `http://individual.utoronto.ca/jordanbell/notes/finitedimdistributions.pdf`

[5] `http://individual.utoronto.ca/jordanbell/notes/markovkernels.pdf`, Theorem 4.

$(X_t)_{t\in I}$ is a stochastic processes whose family of finite-dimensional distributions is $(P_J)_{J\in\mathscr{K}}$, i.e. for $J \in \mathscr{K}$ with elements $t_1 < \cdots < t_n$ and for $B \in \mathscr{E}^J$,

$$
\begin{aligned}
&((X_{t_1} \otimes \cdots \otimes X_{t_n})_* P^\mu)(B) \\
&= \int_E \int_E \cdots \int_E 1_B(x_1, \ldots, x_n) d\nu_{t_n - t_{n-1}}^{x_{n-1}}(x_n) \cdots d\nu_{t_1}^{x_0}(x_1) d\mu(x_0).
\end{aligned}
$$

Applying this with $\mu = \delta_x$ yields

$$
((X_{t_1} \otimes \cdots \otimes X_{t_n})_* P^{\delta_x})(B) = \int_E \cdots \int_E 1_B(x_1, \ldots, x_n) d\nu_{t_n - t_{n-1}}^{x_{n-1}}(x_n) \cdots d\nu_{t_1}^{x}(x_1),
$$

and thus, for any $\mu \in \mathscr{P}(E)$,

$$
\begin{aligned}
&\int_E \int_E \cdots \int_E 1_B(x_1, \ldots, x_n) d\nu_{t_n - t_{n-1}}^{x_{n-1}}(x_n) \cdots d\nu_{t_1}^{x}(x_1) d\mu(x) \\
&= \int_E ((X_{t_1} \otimes \cdots \otimes X_{t_n})_* P^{\delta_x})(B) d\mu(x).
\end{aligned}
$$

That is, for $\mu \in \mathscr{P}(E)$, for $J \in \mathscr{K}$, and $B \in \mathscr{E}^J$,

$$
(X_{J*} P^\mu)(B) = \int_E (X_{J*} P^{\delta_x})(B) d\mu(x). \tag{7}
$$

For $J \in \mathscr{K}$, $A_t \in \mathscr{E}$ for $t \in J$, and $A = \prod_{t\in J} A_t \times \prod_{t\in I\setminus J} E \in \mathscr{F} = \mathscr{E}^I$, namely A is a **cylinder set**,[6] let $B = \pi_{I,J}(A) = \prod_{t\in J} A_t \in \mathscr{E}^J$,

$$
X_J^{-1}(B) = \pi_{I,J}^{-1}(B) = A,
$$

so by (7),

$$
P^\mu(A) = \int_E P^{\delta_x}(A) d\mu(x). \tag{8}
$$

Because this is true for all cylinder sets in the product σ-algebra $\mathscr{E}^I$ and $\mathscr{E}^I$ is generated by the collection of cylinder sets,[7] (8) is true for all $A \in \mathscr{F}$.

Let $J \in \mathscr{K}$, with elements $t_1 < \cdots < t_n$, and let $\sigma_n : E^{n+1} \to E^n$ be

$$
\sigma_n(x_0, x_1, \ldots, x_n) = (x_0 + x_1, x_0 + x_1 + x_2, \ldots, x_0 + x_1 + x_2 + \cdots + x_n).
$$

For $B \in \mathscr{E}^n$ using (1) we obtain by induction

$$
\begin{aligned}
&\int_E \int_E \cdots \int_E 1_B(x_1, \ldots, x_{n-1}, x_n) d\nu_{t_n - t_{n-1}}^{x_{n-1}}(x_n) \cdots d\nu_{t_1}^{x_0}(x_1) d\mu(x_0) \\
&= \int_E \int_E \cdots \int_E 1_B(x_1, \ldots, x_{n-1}, x_n + x_{n-1}) d\nu_{t_n - t_{n-1}}(x_n) \cdots d\nu_{t_1}^{x_0}(x_1) d\mu(x_0) \\
&= \cdots \\
&= \int_E \int_E \cdots \int_E 1_B \circ \sigma_n d\nu_{t_n - t_{n-1}}(x_n) \cdots d\nu_{t_1}(x_1) d\mu(x_0).
\end{aligned}
$$

[6]See http://individual.utoronto.ca/jordanbell/notes/productmeasure.pdf
[7]See http://individual.utoronto.ca/jordanbell/notes/productmeasure.pdf

Thus, with P_J the probability measure on $\mathscr{E}^J$ defined in (5),

$$\int_{E^J} 1_B dP_J = P_J(B) = \int_E \int_E \cdots \int_E 1_B \circ \sigma_n d\nu_{t_n - t_{n-1}}(x_n) \cdots d\nu_{t_1}(x_1) d\mu(x_0).$$

For $f : E^n \to [0, \infty]$ a Borel measurable function, there is a sequence of measurable simple functions pointwise increasing to f, and applying the monotone convergence theorem yields

$$\int_{E^n} f dP_J = \int_E \int_E \cdots \int_E f \circ \sigma_n d\nu_{t_n - t_{n-1}}(x_n) \cdots d\nu_{t_1}(x_1) d\mu(x_0). \tag{9}$$

2 Increments

Let $(\Omega, \mathscr{F}, P, (X_t)_{t \in I})$ be a stochastic process with state space E. X is said to have **stationary increments** if there is a family $(\nu_t)_{t \in I}$ of probability measures on $\mathscr{E}$ such that for all $s, t \in I$ with $s \leq t$,

$$P_*(X_t - X_s) = \nu_{t-s}.$$

In particular, for $s = t$ this implies that $P_*(0) = \nu_0$, hence $\nu_0 = \delta_0$.

A stochastic process is said to have **independent increments** if for any $J \in \mathscr{K}$, with elements $0 = t_0 < t_1 < \cdots < t_n$, the random variables

$$X_{t_0}, \quad X_{t_1} - X_{t_0}, \quad \ldots, X_{t_n} - X_{t_{n-1}}$$

are independent.

We now prove that the canonical process associated with the projective family of probability measures induced by a convolution semigroup and any initial distribution has stationary and independent increments.[8]

Theorem 1. *Let $(\nu_t)_{t \in I}$ be a convolution semigroup, let $(P_J)_{J \in \mathscr{K}}$ be the family of measures induced by this convolution semigroup, let $\mu \in \mathscr{P}(E)$, and let $(\Omega, \mathscr{F}, P^\mu, (X_t)_{t \in I})$, $\Omega = E^I$ and $\mathscr{F} = \mathscr{E}^I$, be the associated canonical process. X has stationary increments,*

$$(X_t - X_s)_* P^\mu = \nu_{t-s}, \qquad s \leq t, \tag{10}$$

and has independent increments.

Proof. $\nu_0 = \delta_0$, so (10) is immediate when $s = t$. When $s < t$, let

$$Y = X_s \otimes X_t = X_{\{s,t\}} = \pi_{I,\{s,t\}},$$

which is measurable $\mathscr{F} \to \mathscr{E} \otimes \mathscr{E}$, and let $q : E \times E \to E$ be $(x_1, x_2) \mapsto x_2 - x_1$, which is continuous and hence Borel measurable. Then $q \circ Y$ is measurable

[8] Heinz Bauer, *Probability Theory*, p. 321, Theorem 37.2.

$\mathscr{F} \to \mathscr{E}$, and for $B \in \mathscr{E}$,

$$\begin{aligned}(q \circ Y)^{-1}(B) &= \{\omega \in \Omega : (q \circ Y)(\omega) \in B\} \\ &= \{\omega \in \Omega : X_t(\omega) - X_s(\omega) \in B\} \\ &= (X_t - X_s)^{-1}(B),\end{aligned}$$

and thus

$$(q \circ Y)_* P^\mu = (X_t - X_s)_* P^\mu. \tag{11}$$

Now, according to (6),

$$Y_* P^\mu = (\pi_{I,\{s,t\}})_* P^\mu = P_{\{s,t\}}.$$

Therefore, using that $x_2 - x_1 \in B$ if and only if $x_2 \in x_1 + B$ and also using $\nu_{t-s}^{x_1}(x_1 + B) = \nu_{t-s}(B)$,

$$\begin{aligned}(X_t - X_s)_* P^\mu(B) &= (q \circ Y)_* P^\mu(B) \\ &= Y_* P^\mu(q^{-1}(B)) \\ &= P_{\{s,t\}_*}(q^{-1}(B)) \\ &= \int_E \int_E \int_E 1_{q^{-1}(B)}(x_1, x_2) d\nu_{t-s}^{x_1}(x_2) d\nu_s^x(x_1) d\mu(x) \\ &= \int_E \int_E \int_E 1_{x_1+B}(x_2) d\nu_{t-s}^{x_1}(x_2) d\nu_s^x(x_1) d\mu(x) \\ &= \nu_{t-s}(B) \int_E \int_E d\nu_s^x(x_1) d\mu(x) \\ &= \nu_{t-s}(B) \int_E \nu_s^x(E) d\mu(x) \\ &= \nu_{t-s}(B) \int_E d\mu(x) \\ &= \nu_{t-s}(B),\end{aligned}$$

which shows that

$$(X_t - X_s)_* P^\mu = \nu_{t-s},$$

and thus that X has stationary increments.

Let $0 = t_0 < t_1 < \cdots < t_n$, let $J = \{t_0, t_1, \ldots, t_n\} \in \mathscr{K}$, write $X_{t_{-1}} = 0$, and let

$$Y_0 = X_{t_0} - X_{t_{-1}}, \quad Y_1 = X_{t_1} - X_{t_0}, \quad \ldots, \quad Y_n = X_{t_n} - X_{t_{n-1}}.$$

For the random variables $Y_0, \ldots, Y_n$ to be independent means for their joint distribution to be equal to the product of the distributions of each, i.e. to prove that X has independent increments, writing

$$Z = Y_0 \otimes \cdots \otimes Y_n = \tau_n \circ (X_{t_0} \otimes \cdots \otimes X_{t_n}) = \tau_n \circ X_J = \tau_n \circ \pi_{I,J},$$

with $\tau_n : E^{n+1} \to E^n$ defined by

$$\tau_n(x_0, x_1, \dots, x_n) = (x_0, x_1 - x_0, \dots, x_n - x_{n-1}),$$

we have to prove that

$$Z_* P^\mu = \prod_{j=0}^n Y_{j*} P^\mu.$$

To prove this, it suffices (because the collection of cylinder sets generates the product σ-algebra) to prove that for any $A_0, \dots, A_n \in \mathscr{E}$ and for $A = \prod_{j=0}^n A_j \in \mathscr{E}^{n+1}$,

$$(Z_* P^\mu)(A) = \left(\prod_{j=0}^n Y_{j*} P^\mu \right)(A),$$

i.e. that

$$(Z_* P^\mu)(A) = \prod_{j=0}^n (Y_{j*} P^\mu)(A_j).$$

We now prove this. Using the change of variables theorem and (6),

$$\begin{aligned}
(Z_* P^\mu)(A) &= \int_{E^{n+1}} 1_A d(Z_* P^\mu) \\
&= \int_\Omega 1_A \circ Z dP^\mu \\
&= \int_\Omega 1_A \circ \tau_n \circ (X_{t_0} \otimes \cdots \otimes X_{t_n}) dP^\mu \\
&= \int_{E^J} 1_A \circ \tau_n d(X_{J*} P^\mu) \\
&= \int_{E^J} 1_A \circ \tau_n dP_J.
\end{aligned}$$

Then applying (9) with $f = 1_A \circ \tau_n$,

$$\begin{aligned}
&\int_{E^J} 1_A \circ \tau_n dP_J \\
&= \int_E \int_E \cdots \int_E 1_A \circ \tau_n \circ \sigma_{n+1} d\nu_{t_n - t_{n-1}}(x_n) \cdots d\nu_{t_0}(x_0) d\mu(x_{-1}) \\
&= \int_E \int_E \cdots \int_E 1_A(x_{-1} + x_0, x_1, \dots, x_n) d\nu_{t_n - t_{n-1}}(x_n) \cdots d\nu_{t_0}(x_0) d\mu(x_{-1}) \\
&= \int_E \int_E \cdots \int_E 1_{A_0}(x_{-1} + x_0) 1_{A_1}(x_1) \cdots 1_{A_n}(x_n) \\
&\quad d\nu_{t_n - t_{n-1}}(x_n) \cdots d\nu_{t_0}(x_0) d\mu(x_{-1}) \\
&= \prod_{j=1}^n \nu_{t_j - t_{j-1}}(A_j) \int_E \int_E 1_{A_0}(x_{-1} + x_0) d\nu_{t_0}(x_0) d\mu(x_{-1}),
\end{aligned}$$

and because $t_0 = 0$ and $\nu_0 = \delta_0$,

$$\begin{aligned}\int_E \int_E 1_{A_0}(x_{-1}+x_0) d\nu_{t_0}(x_0) d\mu(x_{-1}) &= \int_E \int_E 1_{A_0}(x_{-1}+x_0) d\delta_0(x_0) d\mu(x_{-1}) \\ &= \int_E 1_{A_0}(x_{-1}) d\mu(x_{-1}) \\ &= \mu(A_0),\end{aligned}$$

and therefore

$$(Z_* P^\mu)(A) = \mu(A_0) \cdot \prod_{j=1}^n \nu_{t_j - t_{j-1}}(A_j).$$

But we have already proved that (10), which tells us that for each j,

$$Y_{j*} P^\mu = (X_{t_j} - X_{t_{j-1}})_* P^\mu = \nu_{t_j - t_{j-1}},$$

and thus

$$(Z_* P^\mu)(A) = \mu(A_0) \cdot \prod_{j=1}^n (Y_{j*} P^\mu)(A_j).$$

But $Y_{0*} P^\mu = X_{0*} P^\mu$ and from (7) we have

$$(X_{0*} P^\mu)(A_0) = \int_E (X_{0*} P^{\delta_x})(A_0) d\mu(x) = \int_E (\pi_{0*} P^{\delta_x})(A_0) d\mu(x),$$

and, from (5),

$$\begin{aligned}(\pi_{0*} P^{\delta_x})(A_0) &= \int_E \int_E 1_{A_0}(x_0) d\nu_0^y(x_0) d\delta_x(y) \\ &= \int_E \int_E 1_{A_0}(x_0) d\delta_y(x_0) d\delta_x(y) \\ &= \int_E 1_{A_0}(y) d\delta_x(y) \\ &= 1_{A_0}(x),\end{aligned}$$

thus

$$(X_{0*} P^\mu)(A_0) = \int_E 1_{A_0}(x) d\mu(x) = \mu(A_0).$$

Therefore

$$(Z_* P^\mu)(A) = (X_{0*} P^\mu)(A_0) \cdot \prod_{j=1}^n (Y_{j*} P^\mu)(A_j) = \prod_{j=0}^n (Y_{j*} P^\mu)(A_j),$$

which completes the proof that X has independent increments. □

3 The Brownian convolution semigroup and Brownian motion

For $a \in \mathbb{R}$ and $\sigma > 0$, let γ_{a,σ^2} be the **Gaussian measure** on $\mathbb{R}$, the probability measure on $\mathbb{R}$ whose density with respect to Lebesgue measure is

$$p(x, a, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right).$$

For $\sigma = 0$, let

$$\gamma_{a,0} = \delta_a.$$

Define for $t \in I$,

$$\nu_t = \prod_{k=1}^{d} \gamma_{0,t},$$

which is an element of $\mathscr{P}(E)$. For $s, t \in I$, we calculate

$$\nu_s * \mu_t = \left(\prod_{k=1}^{d} \gamma_{0,s}\right) * \left(\prod_{k=1}^{d} \gamma_{0,t}\right) = \prod_{k=1}^{d} (\gamma_{0,s} * \gamma_{0,t}) = \prod_{k=1}^{d} \gamma_{0,s+t} = \nu_{s+t},$$

showing that $(\nu_t)_{t\in I}$ is a convolution semigroup. It is proved using Lévy's continuity theorem that $t \mapsto \nu_t$ is continuous $I \to \mathscr{P}(E)$, showing that $(\nu_t)_{t\in I}$ is a continuous convolution semigroup.[9]

We first prove a lemma (which is made explicit in **Isserlis's theorem**) about the moments of random variables with Gaussian distributions.[10]

Lemma 2. *If $Z : \Omega \to E$ is a random variable with Gaussian distribution ν_τ, $\tau > 0$, then for each n there is some $C_n > 0$ such that*

$$E(|Z|^{2n}) = C_n \tau^n.$$

In particular, $C_2 = d$ and $C_4 = d(d+2)$.

Proof. That Z has distribution ν_τ means that

$$Z_* P = \nu_\tau = \prod_{j=1}^{d} \gamma_{0,\tau}.$$

Write $Z = Z_1 \otimes \cdots \otimes Z_d$, each of which has distribution $\gamma_{0,\tau}$, and $Z_* P = \prod_{j=1}^{d} Z_{j*} P$, which means that $Z_1, \ldots, Z_d$ independent. Let $U_j = \tau^{-1/2} Z_j$ for $j = 1, \ldots, d$, and then $U_1, \ldots, U_d$ are independent random variables each with

[9] `http://individual.utoronto.ca/jordanbell/notes/markovkernels.pdf`, §6.

[10] Heinz Bauer, *Probability Theory*, p. 341, Lemma 40.2.

distribution $\gamma_{0,1}$. Then using the multinomial formula,

$$\begin{aligned} E(|Z|^{2n}) &= E((Z_1^2 + \cdots + Z_d^2)^n) \\ &= \tau^n \cdot E((U_1^2 + \cdots + U_d^2)^n) \\ &= \tau^n \cdot E\left(\sum_{k_1+\cdots+k_d=n} \frac{n!}{k_1! \cdots k_d!} \prod_{1\le i\le d} U_j^{2k_i} \right) \\ &= \tau^n \cdot \sum_{k_1+\cdots+k_d=n} \frac{n!}{k_1!\cdots k_d!} E\left(\prod_{1\le i\le d} U_j^{2k_i} \right). \end{aligned}$$

For $n = 2$, since $E(U_iU_j) = E(U_i)E(U_j) = 0$ for $i \neq j$,

$$\tau^2 \cdot \sum_{k_1+\cdots+k_d=2} \frac{2}{k_1!\cdots k_d!} E\left(\prod_{1\le i\le d} U_i^{2k_i} \right) = \tau^2 \cdot \sum_{j=1}^d E(U_j^2) = \tau^2 \cdot \sum_{j=1}^d 1 = d\tau^2,$$

showing that $C_2 = d$. □

A stochastic process $(\Omega, \mathscr{F}, P, (X_t)_{t\in I})$ with state space E is called **a d-dimensional Brownian motion** when:

1. For $s \le t$,
$$(X_t - X_s)_* P = \nu_{t-s},$$
and thus X has stationary increments.

2. X has independent increments.

3. For almost all $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ is continuous $I \to E$.

We call $X_{0*}P$ the **initial distribution** of the Brownian motion. When $X_{0*}P = \delta_x$ for some $x \in E$, we say that x **is the starting point of the Brownian motion**. We now prove that for any Borel probability measure on $\mathscr{E}$, in particular δ_x, there is a d-dimensional Brownian motion which has this as its initial distribution.[11]

Theorem 3 (Brownian motion)**.** *For any $\mu \in \mathscr{P}(E)$, there is a d-dimensional Brownian motion with initial distribution μ.*

Proof. Let $(P_J)_{J\in\mathscr{K}}$ be the family of measures induced by the Brownian convolution semigroup

$$\nu_t = \prod_{k=1}^d \gamma_{0,t}, \qquad t \in I,$$

and let $(\Omega, \mathscr{F}, P^\mu, (X_t)_{t\in I})$, $\Omega = E^I$ and $\mathscr{F} = \mathscr{E}^I$, be the associated canonical process. Theorem 1 tells us that X has stationary increments,

$$(X_t - X_s)_* P^\mu = \nu_{t-s}, \qquad s \le t, \tag{12}$$

[11] Heinz Bauer, *Probability Theory*, p. 342, Theorem 40.3.

and has independent increments. For $\tau = t - s > 0$, by (12) and Lemma 2,

$$E(|X_t - X_s|^4) = d(d+2)\tau^2 = d(d+2)|t-s|^2.$$

Because $E(|X_t - X_t|^4) = E(0) = 0$, we have that for any $s, t \in I$,

$$E(|X_t - X_s|^4) = d(d+2)|t-s|^2.$$

The initial distribution of X is $X_{0*}P^\mu = \mu$. For $\alpha = 4, \beta = 1, c = d(d+2)$, the **Kolmogorov continuity theorem**[12] tells us that there is a continuous modification B of X. That is, there is a stochastic process $(B_t)_{t \in I}$ such that for each $\omega \in \Omega$, the path $t \mapsto B_t(\omega)$ is continuous $I \to E$, namely, B is a **continuous stochastic process**, and for each $t \in I$,

$$P(X_t = B_t) = 1,$$

namely, B is a **modification** of X. Because B is a modification of X, B has the same finite-dimensional distributions as X,[13] from which it follows that B satisfies (12) and has independent increments. For $A \in \mathscr{E}$, because B is a modification of X,

$$(B_{0*}P^\mu)(A) = P^\mu(B_0 \in A) = P^\mu(X_0 \in A) = (X_{0*}P^\mu)(A),$$

thus $B_{0*}P^\mu = X_{0*}P^\mu = \mu$, namely, B has initial distribution μ. Therefore, B is a Brownian motion (indeed, all the paths of B are continuous, not merely almost all of them) that has initial distribution μ, proving the claim. □

For $\mu \in \mathscr{P}(E)$, let $(\Omega, \mathscr{F}, P^\mu, (B_t)_{t \in I})$ be the d-dimensional Brownian motion with initial distribution μ constructed in Theorem 3; we are not merely speaking about some d-dimensional Brownian motion but about this construction, for which $\Omega = E^I$, all whose paths are continuous rather than merely almost all whose paths are continuous. For a measurable space $(A, \mathscr{A})$ and topological spaces X and Y, a function $f : X \times A \to Y$ is called a **Carathéodory function** if for each $x \in X$, the map $a \mapsto f(x, a)$ is measurable $\mathscr{A} \to \mathscr{B}_Y$, and for each $a \in A$, the map $x \mapsto f(x, a)$ is continuous $X \to Y$. It is a fact[14] that if X is a separable metrizable space and Y is a metrizable space, then any Carathéodory function $f : X \times A \to Y$ is measurable $\mathscr{B}_X \otimes \mathscr{A} \to \mathscr{B}_Y$, namely it is **jointly measurable**. $B : I \times \Omega \to E$ is a Carathéodory function. $I = \mathbb{R}_{\geq 0}$, with the subspace topology inherited from $\mathbb{R}$, is a separable metrizable space, and $E = \mathbb{R}^d$ is a metrizable space, and therefore the d-dimensional Brownian motion B is jointly measurable.

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[12] `http://individual.utoronto.ca/jordanbell/notes/kolmogorovcontinuity.pdf`, Theorem 2.

[13] `http://individual.utoronto.ca/jordanbell/notes/kolmogorovcontinuity.pdf`, Lemma 1.

[14] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 153, Lemma 4.51.

The **Kolmogorov-Chentsov theorem**[15] says that if a stochastic process $(X_t)_{t\in I}$ with state space E satisfies, for $\alpha, \beta, c > 0$,

$$E(|X_s - X_s|^\alpha) \leq c|t-s|^{1+\beta}, \qquad s, t \in I,$$

and almost every path of X is continuous, then for almost every $\omega \in \Omega$, for every $0 < \gamma < \frac{\beta}{\alpha}$ the map $t \mapsto X_t(\omega)$ is **locally γ-Hölder continuous**: for each $t_0 \in I$ there is some $0 < \epsilon_{t_0} < 1$ and some C_{t_0} such that

$$|X_t(\omega) - X_s(\omega)| \leq C_{t_0}|t-s|^\gamma, \qquad |s - t_0| < \epsilon_{t_0}, |t - t_0| < \epsilon_{t_0}.$$

For $\mu \in \mathscr{P}(E)$, let $(\Omega, \mathscr{F}, P^\mu, (B_t)_{t\in I})$ be the d-dimensional Brownian motion with initial distribution μ formed in Theorem 3. For $s \leq t$, $(B_t - B_s)_* P^\mu = \nu_{t-s}$, and thus Lemma 2 tells us that for each $n \geq 1$ there is some C_n with which $E(|B_t - B_s|^{2n}) = C_n(t-s)^n$ for all $s < t$. Then $E(|B_t - B_s|^{2n}) \leq C_n|t-s|^n$ for all $s, t \in I$. For $n > 1$ and for $\alpha_n = 2n$ and $\beta_n = n - 1$,

$$\frac{\beta_n}{\alpha_n} = \frac{n-1}{2n} = \frac{1}{2} - \frac{1}{2n},$$

and for $n > 2$, take some $\frac{\beta_{n-1}}{\alpha_{n-1}} < \gamma_n < \frac{\beta_n}{\alpha_n}$. Let N_n be the set of those $\omega \in \Omega$ for which $t \mapsto B_t(\omega)$ is not locally γ_n-Hölder continuous. Then the Kolmogorov-Chentsov theorem yields $P^\mu(N_n) = 0$. Let $N = \bigcup_{n>2} N_n$, which is a P^μ-null set. For $\omega \in \Omega \setminus N$ and for any $0 < \gamma < \frac{1}{2}$, there is some γ_n satisfying $\gamma \leq \gamma_n < \frac{1}{2}$, and hence the map $t \mapsto B_t(\omega)$ is locally γ_n-Hölder continuous, which implies that this map is locally γ-Hölder continuous.[16] We summarize what we have just said in the following theorem.

Theorem 4. *Let $\mu \in \mathscr{P}(E)$ and let $(\Omega, \mathscr{F}, P^\mu, (B_t)_{t\in I})$ be the d-dimensional Brownian motion with initial distribution μ formed in Theorem 3. For almost all $\omega \in \Omega$, for all $0 < \gamma < \frac{1}{2}$, the map $t \mapsto B_t(\omega)$ is locally γ-Hölder continuous.*

4 Lévy processes

A stochastic process $(X_t)_{t\in I}$ with state space E is called a **Lévy process**[17] if (i) $X_0 = 0$ almost surely, (ii) X has stationary and independent increments, and (iii) for any $a > 0$,

$$\lim_{t\downarrow 0} P(|X_t| \geq \epsilon) = 0.$$

Because $X_0 = 0$ almost surely and X has stationary increments, (iii) yields for any $t \in I$,

$$\lim_{s\to t} P(|X_s - X_s| \geq \epsilon) = 0. \tag{13}$$

[15]http://individual.utoronto.ca/jordanbell/notes/kolmogorovcontinuity.pdf, Theorem 4.

[16]http://individual.utoronto.ca/jordanbell/notes/kolmogorovcontinuity.pdf, Lemma 3.

[17]See David Applebaum, *Lévy Processes and Stochastic Calculus*, p. 39, §1.3.

In any case, (13) is sufficient for (iii) to be true. Moreover, (iii) means that $X_s \to X_t$ in the **topology of convergence in probability** as $s \to t$, and if $X_s \to X_t$ almost surely then $X_s \to X_t$ in the topology of convergence in probability; this is proved using Egorov's theorem.[18] Thus, a d-dimensioanl Brownian motion with starting point 0 is a Lévy process; we do not merely assert that the Brownian motion formed in Theorem 3 is a Lévy process. There is much that can be said generally about Lévy processes, and thus the fact that any d-dimensional Brownian motion with starting point 0 is a Lévy process lets us work in a more general setting in which some results may be more naturally proved: if we work merely with a Lévy process we know less about the process and thus have less open moves.

[18] `http://individual.utoronto.ca/jordanbell/notes/L0.pdf`, Theorem 3.