The Cameron-Martin theorem

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1 Gaussian vectors in a Hilbert space

Lemma 1. *Let* $(\Omega, \mathfrak{F})$ *be a measurable space and let* (Y, d) *be a metric space. Suppose that* (f_n) *is a sequence of measurable functions* $(\Omega, \mathfrak{F}) \to (Y, \mathfrak{B}_Y)$, $A \in \mathfrak{F}$, $y_0 \in Y$, *and* $f_n(\omega)$ *converges in* Y *for all* $\omega \in A$. *Then* $f : \Omega \to Y$ *defined by*

$$f(\omega) = \begin{cases} \lim_{n\to\infty} f_n(\omega) & \omega \in A \\ y_0 & \omega \notin A \end{cases}$$

is measurable.

Proof. Because the Borel σ-algebra $\mathfrak{B}_Y$ is generated by the collection of closed sets in Y, it suffices to prove that $f^{-1}(F) \in \mathfrak{F}$ when F is a closed set in Y. Let

$$G_n = \left\{ y \in Y : d(y, F) < \frac{1}{n} \right\}.$$

Because $y \mapsto d(y, F)$ is continuous, each G_n is open. Because F is closed, $F = \bigcap_{n=1}^{\infty} G_n$.

If $\omega \in A \cap f^{-1}(F)$ and $k \geq 1$, then because G_k is an open neighborhood of $f(\omega)$ and $f_n(\omega) \to f(\omega) \in G_k$, there is some m_k such that for $n \geq m_k$ the point $f_n(\omega)$ belongs to G_k. Thus

$$A \cap f^{-1}(F) \subset A \cap \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} f_n^{-1}(G_k).$$

On the other hand, if ω belongs to the right-hand side then for each k there is some m_k such that for $n \geq m_k$, $f_n(\omega) \in G_k$. Because $f_n(\omega) \to f(\omega)$, this means that $f(\omega) \in \overline{G_k}$. This is true for all k, so $f(\omega) \in \bigcap_{k=1}^{\infty} \overline{G_k}$, and because $\overline{G_{k+1}} \subset G_k$, it is the case that $f(\omega) \subset \bigcap_{k=1}^{\infty} G_k = F$. Therefore,

$$A \cap f^{-1}(F) = A \cap \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} f_n^{-1}(G_k),$$

which shows that $A \cap f^{-1}(F) \in \mathfrak{F}$. If $y_0 \in F$, then $f^{-1}(F) = A^c \cup (A \cap f^{-1}(F)) \in \mathfrak{F}$, and if $y_0 \notin F$, then $f^{-1}(F) = A \cap f^{-1}(F) \in \mathfrak{F}$. Therefore f is measurable. □

Let $\mathscr{H}$ be a separable real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let (e_j) be an orthonormal basis for $\mathscr{H}$. Let (ξ_j) be a sequence of independent random variables $(\Omega, \mathfrak{F}, \mathbb{P}) \to (\mathbb{R}, \mathfrak{B}_\mathbb{R})$ with distribution $(\xi_j)_* \mathbb{P} = \gamma_1$, where γ_{σ^2} is the Gaussian measure on $\mathbb{R}$ with variance σ^2.[1] Let (σ_j) be a sequence of nonnegative real numbers satisfying $\sum_{j=1}^\infty \sigma_j^2 < \infty$. Define $X_n : \Omega \to \mathscr{H}$ by

$$X_n(\omega) = \sum_{j=1}^n \sigma_j \xi_j(\omega) e_j,$$

which is measurable $(\Omega, \mathfrak{F}) \to (\mathscr{H}, \mathfrak{B}_\mathscr{H})$. For $X_n(\omega)$ to be a Cauchy sequence in $\mathscr{H}$, it is necessary and sufficient that $\sum_{j=1}^\infty |\sigma_j \xi_j(\omega)|^2 < \infty$.[2] But

$$\sum_{j=1}^\infty \mathbb{E}|\sigma_j \xi_j|^2 = \sum_{j=1}^\infty \sigma_j^2 \mathbb{E}|\xi_j|^2 = \sum_{j=1}^\infty \sigma_j^2 < \infty$$

implies that the series $\sum_{j=1}^\infty |\sigma_j \xi_j|^2$ is convergent almost surely: for some $A \in \mathfrak{F}$ with $\mathbb{P}(A) = 1$ the series $\sum_{j=1}^\infty |\sigma_j \xi_j(\omega)|^2$ converges for $\omega \in A$. For $\omega \in A$ we define $X(\omega) \in \mathscr{H}$ to be the limit of the Cauchy sequence $X_n(\omega)$,

$$X(\omega) = \sum_{j=1}^\infty \sigma_j \xi_j(\omega) e_j, \tag{1}$$

and otherwise we define $X(\omega) = 0$. By Lemma 1, X is measurable $(\Omega, \mathfrak{F}) \to (\mathscr{H}, \mathfrak{B}_\mathscr{H})$.[3]

For X defined in (1) and for $f \in \mathscr{H}$ with

$$f = \sum_j \langle f, e_j \rangle e_j = \sum_j f_j e_j,$$

we have for $\omega \in A$,

$$\langle f, X \rangle = \sum_{j=1}^\infty f_j \sigma_j \xi_j(\omega).$$

This satisfies

$$\mathbb{E} \langle X, f \rangle = 0,$$

[1] http://individual.utoronto.ca/jordanbell/notes/gaussian.pdf

[2] http://individual.utoronto.ca/jordanbell/notes/parseval.pdf

[3] Mikhail Lifshits, *Lectures on Gaussian Processes*, p. 7, Example 2.2; Lifshits calls this a **Karhunen-Loève expansion** of X.

and for $f, g \in \mathscr{H}$,

$$\begin{aligned}\operatorname{Cov}(\langle X, f\rangle, \langle X, g\rangle) &= E(\langle X, f\rangle \cdot \langle X, g\rangle) \\ &= E\left(\sum_{j=1}^{\infty} \sigma_j f_j \xi_j \cdot \sum_{k=1}^{\infty} \sigma_k g_k \xi_k\right) \\ &= \sum_{j=1}^{\infty} \sigma_j^2 f_j g_j.\end{aligned}$$

Define $K : \mathscr{H} \to \mathscr{H}$ by

$$Ke_j = \sigma_j^2 e_j,$$

which is a Hilbert-Schmidt operator.[4] It satisfies

$$\langle Kf, g\rangle = \operatorname{Cov}(\langle X, f\rangle, \langle X, g\rangle).$$

2 Wiener measure

Let $\mathscr{X} = C[0,1]$, which is a separable Banach space with the supremum norm, whose dual space $\mathscr{X}^*$ is the signed measures of bounded variation on $[0,1]$.[5] For $\mu \in \mathscr{X}^*$ and $f \in \mathscr{X}$, write

$$\langle f, \mu\rangle = \int_{[0,1]} f d\mu.$$

Let $W \in \mathscr{X}^*$ be Wiener measure on $\mathscr{X}$, define $B_t f = f(t)$, and define $B : \mathscr{X} \to \mathscr{X}$ by $Bf = f$.[6] The stochastic process $(B_t)_{t\in[0,1]}$ is a Brownian motion. For $s, t \in [0,1]$,

$$\mathbb{E}B_t = 0, \qquad \operatorname{Cov}(B_s, B_t) = \mathbb{E}(B_s B_t) = \min(s,t).$$

$B : (\mathscr{X}, \mathfrak{B}_{\mathscr{X}}) \to (\mathscr{X}, \mathfrak{B}_{\mathscr{X}})$ is measurable, and $B_* W = W$, i.e. the distribution of B is Wiener measure. For $\mu \in \mathscr{X}^*$,

$$\mathbb{E}\langle B, \mu\rangle = \mathbb{E}\int_{[0,1]} B_t d\mu(t) = \int_{[0,1]} \mathbb{E}B_t d\mu = 0$$

and for $\mu, \nu \in \mathscr{X}^*$,

$$\begin{aligned}\operatorname{Cov}(\langle B, \mu\rangle, \langle B, \nu\rangle) &= \mathbb{E}\left(\int_{[0,1]} B_s d\mu(s) \cdot \int_{[0,1]} B_t d\nu(t)\right) \\ &= \int_{[0,1]\times[0,1]} \mathbb{E}(B_s B_t) d\mu(s) d\nu(t) \\ &= \int_{[0,1]\times[0,1]} \min(s,t) d\mu(s) d\nu(t).\end{aligned}$$

[4] `http://individual.utoronto.ca/jordanbell/notes/traceclass.pdf`
[5] `http://individual.utoronto.ca/jordanbell/notes/CK.pdf`
[6] `http://individual.utoronto.ca/jordanbell/notes/donsker.pdf`

Define $K : \mathscr{X}^* \to \mathscr{X}$ by

$$(K\mu)(t) = \int_{[0,1]} \min(s,t) d\mu(s),$$

which satisfies

$$\mathrm{Cov}(\langle B, \mu \rangle, \langle B, \nu \rangle) = \langle K\mu, \nu \rangle .$$

3 Measurable linear functionals

Let $\mathscr{X}$ be a Fréchet space with dual space $\mathscr{X}^*$, and for $f \in \mathscr{X}$ and $\mu \in \mathscr{X}^*$ denote the dual pairing by

$$\langle f, \mu \rangle ,$$

and we also use this notation when μ is a function $\mathscr{X} \to \mathbb{R}$ that need not belong to $\mathscr{X}^*$. Suppose that $X : (\Omega, \mathfrak{F}, \mathbb{P}) \to (\mathscr{X}, \mathfrak{B}_{\mathscr{X}})$ is measurable, and that X is Gaussian with $\mathbb{E}X = 0 \in \mathscr{X}$ and covariance $K : \mathscr{X}^* \to \mathscr{X}$. That is,

$$\mathbb{E} \langle X, \mu \rangle = \langle 0, \mu \rangle = 0$$

for all $\mu \in \mathscr{X}^*$, and $K : \mathscr{X}^* \to \mathscr{X}$ is a continuous linear operator satisfying

$$\mathbb{E}(\langle X, \mu \rangle \cdot \langle X, \nu \rangle) = \mathrm{Cov}(\langle X, \mu \rangle, \langle X, \nu \rangle) = \langle K\mu, \nu \rangle$$

for all $\mu, \nu \in \mathscr{X}^*$. Let $P = X_* \mathbb{P}$ be the distribution of X; P is a Borel probability measure on $\mathscr{X}$.

For $\mu \in \mathscr{X}^*$, by the change of variables formula,

$$\mathbb{E}| \langle X, \mu \rangle |^2 = \int_\Omega | \langle X(\omega), \mu \rangle |^2 d\mathbb{P}(\omega) = \int_{\mathscr{X}} | \langle f, \mu \rangle |^2 dP(f) = \int_{\mathscr{X}} |\mu|^2 dP.$$

Let $J : \mathscr{X}^* \to L^2(\mathscr{X}, P)$ be the embedding, and let $\mathscr{X}_P^*$ be the closure of $J(\mathscr{X}^*)$ in $L^2(\mathscr{X}, P)$. Thus $\mathscr{X}_P^*$ is a Hilbert space with the inner product

$$\langle \phi, \psi \rangle_{\mathscr{X}_P^*} = \int_{\mathscr{X}} \phi \cdot \psi dP = \mathbb{E}(\langle X, \phi \rangle \cdot \langle X, \psi \rangle).$$

Elements of $\mathscr{X}_P^*$ are called **measurable linear functionals**; elements of $\mathscr{X}^*$ are continuous linear functionals.

$J : \mathscr{X}^* \to \mathscr{X}_P^*$ is a continuous linear map, and there is a unique continuous linear map $I : (\mathscr{X}_P^*)^* \to \mathscr{X}$ satisfying

$$\langle I\phi, \mu \rangle = \langle \phi, J\mu \rangle_{\mathscr{X}_P^*} = \langle \phi, \mu \rangle_{\mathscr{X}_P^*} = \mathbb{E}(\langle X, \phi \rangle \cdot \langle X, \mu \rangle)$$

for $\phi \in (\mathscr{X}_P^*)^* = \mathscr{X}_P^*$ and $\mu \in \mathscr{X}^*$. If $I\phi = 0$ then

$$\langle \phi, J\mu \rangle_{\mathscr{X}_P^*} = \langle I\phi, \mu \rangle = \langle 0, \mu \rangle = 0$$

for all $\mu \in \mathscr{X}^*$. Let $\mu_n \in \mathscr{X}^*$ with $J\mu_n \to \phi$ in $L^2(\mathscr{X}, P)$. Then $\langle \phi, J\mu_n \rangle_{\mathscr{X}_P^*} \to \langle \phi, \phi \rangle_{\mathscr{X}_P^*}$, and because each $\langle \phi, J\mu_n \rangle_{\mathscr{X}_P^*} = 0$ we get that $\langle \phi, \phi \rangle_{\mathscr{X}_P^*} = 0$, which means that $\phi = 0$. Therefore I is injective.

We have assumed that X has covariance $K : \mathscr{X}^* \to \mathscr{X}$, which means that for $\mu, \nu \in \mathscr{X}^*$,

$$\langle K\mu, \nu \rangle = \mathbb{E}(\langle X, \mu \rangle \cdot \langle X, \nu \rangle).$$

For $\mu, \nu \in \mathscr{X}^*$,

$$\langle IJ\mu, \nu \rangle = \langle J\mu, J\nu \rangle_{\mathscr{X}_P^*} = \mathbb{E}(\langle X, \mu \rangle \cdot \langle X, \nu \rangle) = \langle K\mu, \nu \rangle,$$

which implies that $K = IJ$.

Let

$$H_P = I\mathscr{X}_P^*,$$

which is a linear subspace of $\mathscr{X}$. For $f, g \in H_P$, let

$$\langle f, g \rangle_{H_P} = \left\langle I^{-1}f, I^{-1}g \right\rangle_{\mathscr{X}_P^*}.$$

4 Examples of H_P

Take $X : \Omega \to \mathscr{H}$ from §1, with $\mathbb{E}X = 0$ and with covariance $K : \mathscr{H} \to \mathscr{H}$ defined by

$$Ke_j = \sigma_j^2 e_j,$$

which is a Hilbert-Schmidt operator satisfying

$$\langle Kf, g \rangle = \mathrm{Cov}(\langle X, f \rangle, \langle X, g \rangle).$$

For $f, g \in \mathscr{H} = \mathscr{H}^*$,

$$\langle Jf, Jg \rangle_{\mathscr{X}_P^*} = \langle Kf, g \rangle = \sum_{j=1}^{\infty} \sigma_j^2 f_j g_j.$$

Check that $\mathscr{X}_P^*$ is the set of those $\phi : \mathscr{H} \to \mathbb{R}$ such that

$$\sum_{j=1}^{\infty} |\langle e_j, \phi \rangle|^2 \sigma_j^2 < \infty.$$

Writing $\phi_j = \langle \phi, e_j \rangle$,

$$\begin{aligned}
\langle I\phi, e_k \rangle &= \langle \phi, Je_k \rangle_{\mathscr{X}_P^*} \\
&= \langle \phi, e_k \rangle_{\mathscr{X}_P^*} \\
&= \mathbb{E}(\langle X, \phi \rangle \cdot \langle X, e_k \rangle) \\
&= \mathbb{E}\left(\sum_{j=1}^{\infty} \phi_j \sigma_j \xi_j \cdot \sigma_k \xi_k \right) \\
&= \sigma_j^2 \phi_j.
\end{aligned}$$

For $\phi \in \mathscr{X}_P^*$, write $h = I\phi$, and then

$$\|h\|_{H_P}^2 = \|\phi\|_{\mathscr{X}_P^*}^2 = \sum_{j=1}^{\infty} \sigma_j^2 \phi_j^2.$$

But

$$\begin{aligned} h_k &= \langle h, e_k \rangle \\ &= \langle I\phi, e_k \rangle = \langle \phi, Je_k \rangle_{\mathscr{X}_P^*} \\ &= \mathbb{E}(\langle X, \phi \rangle \cdot \langle X, e_k \rangle) \\ &= \mathbb{E}\left(\sum_{j=1}^{\infty} \phi_j \sigma_j \xi_j \cdot \sigma_k \xi_k \right) \\ &= \sigma_k^2 \phi_k, \end{aligned}$$

so

$$\|h\|_{H_P}^2 = \sum_{j=1}^{\infty} \frac{h_j^2}{\sigma_j^2}.$$

We then check that

$$H_P = \left\{ h \in \mathscr{H} : \sum_{j=1}^{\infty} \frac{h_j^2}{\sigma_j^2} < \infty \right\}.$$

Now take $\mathscr{X} = \mathbb{R}^d$ and let $X : (\Omega, \mathfrak{F}, \mathbb{P}) \to (\mathbb{R}^d, \mathscr{B}_{\mathbb{R}}^d)$ be a random vector that is Gaussian with $\mathbb{E}X = 0$ and positive-definite covariance $K : \mathbb{R}^d \to \mathbb{R}^d$, and let $P = X_*\mathbb{P}$, a Borel probability measure on $\mathbb{R}^d$. Because K is a positive-definite symmetric matrix, by the spectral theorem there is an orthonormal basis $e_1, \ldots, e_d$ for $\mathbb{R}^d$ and positive real numbers $\sigma_1, \ldots, \sigma_d$ such that $Ke_j = \sigma_j^2 e_j$. For almost all $\omega \in \Omega$,

$$X(\omega) = \sum_{j=1}^{d} \sigma_j \xi_j(\omega) e_j.$$

From our work before, $H_P = \mathbb{R}^d$, and

$$\langle f, g \rangle_{H_P} = \sum_{j=1}^{\infty} \frac{f_j g_j}{\sigma_j^2} = \left\langle K^{-1} f, g \right\rangle.$$

5 The factorization theorem

The following is proved in Lifshits, and there is called the **factorization theorem**.[7]

[7] Mikhail Lifshits, *Lectures on Gaussian Processes*, p. 26, Theorem 4.1.

Theorem 2 (Factorization theorem)**.** *If $\mathscr{X}$ is a Fréchet space, $\mathscr{H}$ is a Hilbert space, and $L : \mathscr{H} \to \mathscr{X}$ is an injective linear map such that*

$$K = LL^*,$$

then

$$H_P = L\mathscr{H}$$

and

$$\langle f, g \rangle_{H_P} = \left\langle L^{-1}f, L^{-1}g \right\rangle_{\mathscr{H}}$$

for all $f, g \in H_P$.

Let $\mathscr{X} = C[0,1]$, from §2. Here, $B : (\mathscr{X}, \mathfrak{B}_{\mathscr{X}}, W) \to (\mathscr{X}, \mathfrak{B}_{\mathscr{X}})$ is $Bf = f$, $P = B_* W = W$, and the covariance of B is $K : \mathscr{X}^* \to \mathscr{X}$,

$$(K\mu)(t) = \int_{[0,1]} \min(s,t) d\mu(s),$$

satisfying

$$\langle K\mu, \nu \rangle = \mathbb{E}(\langle B, \mu \rangle, \langle B, \nu \rangle).$$

Take $\mathscr{H} = L^2[0,1]$, with Lebesgue measure. Define $L : \mathscr{H} \to \mathscr{X}$ by

$$(Lf)(t) = \int_0^t f(s) ds.$$

Indeed, Lf is continuous, and L is linear and injective. $\mathscr{X}^*$ is the signed measures of bounded variation on $[0,1]$. For $\mu \in \mathscr{X}^*$, Fubini's theorem yields

$$\begin{aligned}
\langle f, L^*\mu \rangle_{\mathscr{H}} &= \langle Lf, \mu \rangle \\
&= \int_{[0,1]} (Lf)(t) d\mu(t) \\
&= \int_{[0,1]} \left(\int_0^t f(s) ds \right) d\mu(t) \\
&= \int_0^1 \left(\int_{[s,1]} d\mu(t) \right) f(s) ds \\
&= \int_0^1 \mu[s,1] f(s) ds \\
&= \langle s \mapsto \mu[s,1], f \rangle_{\mathscr{H}} .
\end{aligned}$$

This shows that $L^* : \mathscr{X}^* \to \mathscr{H}$ is

$$(L^*\mu)(s) = \mu[s,1].$$

For $\mu \in \mathscr{X}^*$,

$$\begin{aligned}
(LL^*\mu)(t) &= \int_0^t (L^*\mu)(s)ds \\
&= \int_0^t \mu[s,1]ds \\
&= \int_0^1 1_{[0,t]}(s)\left(\int_{[s,1]} d\mu(r)\right) ds \\
&= \int_{[0,1]} \left(\int_0^r 1_{[0,t]}(s)ds\right) d\mu(r) \\
&= \int_{[0,1]} \min(r,t) d\mu(r),
\end{aligned}$$

showing that $LL^* = K$. Then by Theorem 2,

$$H_W = L\mathscr{H}$$

and

$$\langle F, G\rangle_{H_W} = \langle L^{-1}F, L^{-1}G\rangle_{\mathscr{H}},$$

for $F, G \in H_W$. This means that if $F \in H_W$ if and only if there is $f \in L^2[0,1]$ such that

$$F(t) = (Lf)(t) = \int_0^t f(s)ds.$$

This is equivalent to F being absolutely continuous, with $F(0) = 0$ and for almost all $t \in [0,1]$, F is differentiable at t, and $F' \in L^2[0,1]$.[8] Thus, H_W is the collection of absolutely continuous functions $F : [0,1] \to \mathbb{R}$ satisfying $F(0) = 0$ and $F' \in L^2[0,1]$. Furthermore,

$$\langle F, G\rangle_{H_W} = \langle L^{-1}F, L^{-1}G\rangle_{L^2[0,1]} = \langle F', G'\rangle_{L^2[0,1]}.$$

6 The Cameron-Martin theorem

Let $\mathscr{X}$ be a Fréchet space and let $X : (\Omega, \mathfrak{F}, \mathbb{P}) \to (\mathscr{X}, \mathfrak{B}_{\mathscr{X}})$ be a random vector with distribution $P = X_*\mathbb{P}$. For $h \in \mathscr{X}$, $X + h$ is a random vector, and we write $P_h = (X+h)_*\mathbb{P}$. For $A \in \mathfrak{B}_{\mathscr{X}}$,

$$P_h(A) = \mathbb{P}(X + h \in A) = \mathbb{P}(X \in A - h) = P(A - h).$$

If P_h is absolutely continuous with respect to P, written $P_h \ll P$, we say that h is an **admissible shift**.

For $\mathscr{X} = \mathbb{R}^d$, let X be a random vector with state space $\mathscr{X}$ and Gaussian distribution with $\mathbb{E}X = 0$ and covariance I_d, namely a random vector on $\mathscr{X}$

[8] `http://individual.utoronto.ca/jordanbell/notes/totalvariation.pdf`

with the standard Gaussian distribution. Let λ_d be Lebesgue measure on $\mathbb{R}^d$. For $P = X_*\mathbb{P}$, which is a standard Gaussian measure on $\mathbb{R}^d$,

$$dP(x) = (2\pi)^{-d/2} e^{-\langle x,x\rangle/2} d\lambda_d(x).$$

That is, the density of P with respect to λ_d is

$$\frac{dP}{d\lambda_d}(x) = (2\pi)^{-d/2} e^{-\langle x,x\rangle/2}.$$

For $h \in \mathbb{R}^d$ and A a Borel set in $\mathbb{R}^d$,

$$\begin{aligned} P_h(A) &= P(A-h) \\ &= \int_{A-h} (2\pi)^{-d/2} e^{-\langle x,x\rangle/2} dx \\ &= \int_A (2\pi)^{-d/2} e^{-\langle y-h,y-h\rangle/2} dy, \end{aligned}$$

which shows that

$$\frac{dP_h}{d\lambda_d}(x) = (2\pi)^{-d/2} e^{-\langle x-h,x-h\rangle/2}.$$

Because $\lambda_d \ll P$, with

$$\frac{d\lambda_d}{dP}(x) = (2\pi)^{d/2} e^{\langle x,x\rangle/2},$$

the chain rule for the Radon-Nikodym derivative yields

$$\frac{dP_h}{dP}(x) = \frac{dP_h}{d\lambda_d}(x) \cdot \frac{d\lambda_d}{dP}(x) = (2\pi)^{-d/2} e^{-\langle x-h,x-h\rangle/2} \cdot (2\pi)^{d/2} e^{\langle x,x\rangle/2},$$

which is

$$\frac{dP_h}{dP}(x) = e^{\langle h,x\rangle - \langle h,h\rangle/2}.$$

We now get to the **Cameron-Martin theorem**.[9]

Theorem 3 (Cameron-Martin theorem)**.** *Let $\mathscr{X}$ be a Fréchet space, let $X : (\Omega, \mathfrak{F}, \mathbb{P}) \to (\mathscr{X}, \mathfrak{B}_{\mathscr{X}})$ be a random vector that is Gaussian with $\mathbb{E}X = 0$ and covariance $K : \mathscr{X}^* \to \mathscr{X}$, and with distribution $P = X_*\mathbb{P}$. In this case, $P_h \ll P$ if and only if $h \in H_P$.*

If $h \in H_P$, then there is some $\phi \in \mathscr{X}_P^*$ such that $L\phi = h$ and

$$\frac{dP_h}{dP}(f) = e^{\langle f,\phi\rangle - \frac{\langle h,h\rangle_{H_P}}{2}}, \qquad f \in \mathscr{X}.$$

We have established that

$$H_W = \{h \in AC[0,1] : h(0) = 0, h' \in L^2[0,1]\}$$

and

$$\|h\|_{H_W}^2 = \int_0^1 |h'(s)|^2 ds, \qquad h \in H_W.$$

For $h = H_W$ let $\phi \in L^2[0,1]$ such that $L\phi = h$, i.e. $\phi = L^{-1}h$ which means $\phi = h'$ in $L^2[0,1]$.

[9] Mikhail Lifshits, *Lectures on Gaussian Processes*, p. 34, Theorem 5.1.