Compact operators on Banach spaces

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1 Introduction

In this note I prove several things about compact linear operators from one Banach space to another, especially from a Banach space to itself. Some of these may things be simpler to prove for compact operators on a Hilbert space, but since often in analysis we deal with compact operators from one Banach space to another, such as from a Sobolev space to an L^p space, and since the proofs here are not absurdly long, I think it's worth the extra time to prove all of this for Banach spaces. The proofs that I give are completely detailed, and one should be able to read them without using a pencil and paper. When I want to use a fact that is not obvious but that I do not wish to prove, I give a precise statement of it, and I verify that its hypotheses are satisfied.

2 Preliminaries

If X and Y are normed spaces, let $\mathscr{B}(X,Y)$ be the set of bounded linear maps $X \to Y$. It is straightforward to check that $\mathscr{B}(X,Y)$ is a normed space with the operator norm

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\|.$$

If X is a normed space and Y is a Banach space, one proves that $\mathscr{B}(X,Y)$ is a Banach space.[1] Let $\mathscr{B}(X) = \mathscr{B}(X,X)$. If X is a Banach space then so is $\mathscr{B}(X)$, and it is straightforward to verify that $\mathscr{B}(X)$ is a Banach algebra.

To say $T \in \mathscr{B}(X)$ is **invertible** means that there is some $S \in \mathscr{B}(X)$ such that $ST = \mathrm{id}_X$ and $TS = \mathrm{id}_X$, and we write $T^{-1} = S$. It follows from the **open mapping theorem** that if $T \in \mathscr{B}(X)$, $\ker T = \{0\}$, and $T(X) = X$, then T is invertible (i.e. if a bounded linear map is bijective then its inverse is also a bounded linear map, where we use the open mapping theorem to show that the inverse is continuous).

[1] Walter Rudin, *Functional Analysis*, second ed., p. 92, Theorem 4.1.

The **spectrum** $\sigma(T)$ of $T \in \mathscr{B}(X)$ is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda \mathrm{id}_X$ is not invertible. If $T - \lambda \mathrm{id}_X$ is not injective, we say that λ is an **eigenvalue** of T, and then there is some nonzero $x \in \ker(T - \lambda \mathrm{id}_X)$, which thus satisfies $Tx = \lambda x$; we call any nonzero element of $\ker(T - \lambda \mathrm{id}_X)$ an **eigenvector** of T. The **point spectrum** of T is the set of eigenvalues of T.

We say that a subset of a topological space is **precompact** if its closure is compact. The **Heine-Borel theorem** states that a subset S of a complete metric space M is precompact if and only if it is **totally bounded**: to be totally bounded means that for every $\epsilon > 0$ there are finitely many points $x_1, \dots, x_r \in S$ such that $S \subseteq \bigcup_{k=1}^{r} B_\epsilon(x_i)$, where $B_\epsilon(x)$ is the open ball of radius ϵ and center x.

If X and Y are Banach spaces and $B_1(0)$ is the open unit ball in X, a linear map $T : X \to Y$ is said to be **compact** if $T(B_1(0))$ is precompact; equivalently, if $T(B_1(0))$ is totally bounded. Check that a linear map $T : X \to Y$ is compact if and only if the image of *every* bounded set is precompact. Thus, if we want to prove that a linear map is compact we can show that the image of the open unit ball is precompact, while if we know that a linear map is compact we can use that the image of every bounded set is precompact. It is straightforward to prove that a compact linear map is bounded. Let $\mathscr{B}_0(X, Y)$ denote the set of compact linear maps $X \to Y$. It does not take long to prove that $\mathscr{B}_0(X)$ is an ideal in the algebra $\mathscr{B}(X)$.

3 Basic facts about compact operators

Theorem 1. *Let X and Y be Banach spaces. If $T : X \to Y$ is linear, then T is compact if and only if $x_n \in X$ being a bounded sequence implies that there is a subsequence $x_{a(n)}$ such that $Tx_{a(n)}$ converges in Y.*

Proof. Suppose that T is compact and let $x_n \in X$ be bounded, with

$$M = \sup_n \|x_n\| < \infty.$$

Let V be the closed ball in X of radius M and center 0. V is bounded in X, so $N = \overline{T(V)}$ is compact in Y. As $Tx_n \in N$, there is some convergent subsequence $Tx_{a(n)}$ that converges to some $y \in N$.

Suppose that if x_n is a bounded sequence in X then there is a subsequence such that $Tx_{a(n)}$ is convergent, let U be the open unit ball in X, and let $y_n \in T(U)$ be a sequence. It is a fact that a subset of a metric space is precompact if and only if every sequence has a subsequence that converges to some element in the space; this is not obvious, but at least we are only taking as given a fact about metric spaces. (What we have asserted is that a set in a metric space is precompact if and only if it is **sequentially precompact**.) As y_n are in the image of T, there is a subsequence such that $y_{a(n)}$ is convergent and this implies that $T(U)$ is precompact, and so T is a compact operator. ☐

Theorem 2. *Let X and Y be Banach spaces. If $T \in \mathscr{B}_0(X, Y)$, then $T(X)$ is separable.*

Proof. Let U_n be the closed ball of radius n in X. As $\overline{T(U_n)}$ is a compact metric space it is separable, and hence $T(U_n)$, a subset of it, is separable too, say with dense subset L_n. We have

$$T(X) = \bigcup_{n=1}^{\infty} T(U_n),$$

and one checks that $\bigcup_{n=1}^{\infty} L_n$ is a dense subset of the right-hand side, showing that $T(X)$ is separable. □

The following theorem gathers some important results about compact operators.[2]

Theorem 3. *Let X and Y be Banach spaces.*

- *If $T \in \mathscr{B}(X,Y)$, T is compact, and $T(X)$ is a closed subset of Y, then $\dim T(X) < \infty$.*
- *$\mathscr{B}_0(X,Y)$ is a closed subspace of $\mathscr{B}(X,Y)$.*
- *If $T \in \mathscr{B}(X)$, T is compact, and $\lambda \neq 0$, then $\dim \ker(T - \lambda \mathrm{id}_X) < \infty$.*
- *If $\dim X = \infty$, $T \in \mathscr{B}(X)$, and T is compact, then $0 \in \sigma(T)$.*

Proof. If $T \in \mathscr{B}(X,Y)$ is compact and $T(X)$ is closed then as a closed subspace of a Banach space, $T(X)$ is itself a Banach space. Of course $T : X \to T(X)$ is surjective, and X is a Banach space so by the open mapping theorem $T : X \to T(X)$ is an open map. Let $Tx \in T(X)$. As T is an open map, $T(B_1(x))$ is open, and hence $\overline{T(B_1(x))}$ is a neighborhood of Tx. But because T is compact and $B_1(x)$ is bounded, $\overline{T(B_1(x))}$ is compact. Hence $\overline{T(B_1(x))}$ is a compact neighborhood of Tx. As every element of $T(X)$ has a compact neighborhood, $T(X)$ is locally compact. But a locally compact topological vector space is finite dimensional, so $\dim T(X) < \infty$.

It is straightforward to check that $\mathscr{B}_0(X,Y)$ is linear subspace of $\mathscr{B}(X,Y)$. Let T be in the closure of $\mathscr{B}_0(X,Y)$ and let U be the open unit ball in X. We wish to show that $T(U)$ is totally bounded. Let $\epsilon > 0$. As T is in the closure of $\mathscr{B}_0(X,Y)$, there is some $S \in \mathscr{B}_0(X,Y)$ with $\|S - T\| < \epsilon$. As S is compact, its image $S(U)$ is totally bounded, so there are finitely many $Sx_1, \ldots, Sx_r \in S(U)$, with $x_1, \ldots, x_r \in U$, such that $S(U) \subseteq \bigcup_{k=1}^{r} B_\epsilon(Sx_k)$. If $x \in U$, then

$$\|Sx - Tx\| = \|(S - T)x\| \leq \|S - T\| \|x\| < \|S - T\| < \epsilon.$$

Let $x \in U$. Then there is some k such that $Sx \in B_\epsilon(Sx_k)$, and

$$\|Tx - Tx_k\| \leq \|Tx - Sx\| + \|Sx - Sx_k\| + \|Sx_k - Tx_k\| < 3\epsilon,$$

so $T(U) \subseteq \bigcup_{k=1}^{r} B_{3\epsilon}(Tx_k)$, showing that $T(U)$ is totally bounded and hence that T is a compact operator.

[2] Walter Rudin, *Functional Analysis*, second ed., p. 104, Theorem 4.18.

If $T \in \mathscr{B}(X)$ is compact and $\lambda \neq 0$, let $Y = \ker(T - \lambda \mathrm{id}_X)$. (If λ is not an eigenvalue of T, then $Y = \{0\}$.) Y is a closed subspace of X, and hence is itself a Banach space. If $y \in Y$ then $Ty = \lambda y \in Y$. Define $S : Y \to Y$ by $Sy = Ty = \lambda y$, and as T is compact so is S. Now we use the hypothesis that $\lambda \neq 0$: if $y \in Y$, then $S(\frac{1}{\lambda} y) = y$, so $S : Y \to Y$ is surjective. We have shown that $S : Y \to Y$ is compact and that $S(Y)$ is a closed subset of Y (as it is equal to Y), and as a closed image of a compact operator is finite dimensional, we obtain $\dim S(Y) < \infty$, i.e. $\dim Y < \infty$.

If $\dim X = \infty$ and $T \in \mathscr{B}(X)$ is compact, suppose by contradiction that $0 \notin \sigma(T)$. So T is invertible, with $TT^{-1} = \mathrm{id}_X$. As $\mathscr{B}_0(X)$ is an ideal in the algebra $\mathscr{B}(X)$, id_X is compact. Of course $\mathrm{id}_X(X) = X$ is a closed subset of X. But we proved that if the image of a compact linear operator is closed then that image is finite dimensional, contradicting $\dim X = \infty$. □

4 Dual spaces

If X is a normed space, let $X^* = \mathscr{B}(X, \mathbb{C})$, the set of bounded linear maps $X \to \mathbb{C}$. X^* is called the **dual space** of X, and is a Banach space since $\mathbb{C}$ is a Banach space. Define $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{C}$ by

$$\langle x, \lambda \rangle = \lambda(x), \qquad x \in X, \lambda \in X^*.$$

This is called the **dual pairing** of X and X^*.

The following theorem gives an expression for the norm of an element of the dual space.[3]

Theorem 4. *If X is a normed space and V is the closed unit ball in X^*, then*

$$\|x\| = \sup_{\lambda \in V} |\langle x, \lambda \rangle|, \qquad x \in X.$$

Proof. It follows from the Hahn-Banach extension theorem that if $x_0 \in X$, then there is some $\lambda_0 \in X^*$ such that $\lambda_0(x_0) = \|x_0\|$ and such that if $x \in X$ then $|\lambda_0(x)| \leq \|x\|$.[4] That is, that there is some $\lambda_0 \in V$ such that $\lambda_0(x_0) = \|x_0\|$. Hence

$$\sup_{\lambda \in V} |\langle x_0, \lambda \rangle| \geq |\langle x_0, \lambda_0 \rangle| = |\lambda_0(x_0)| = \|x_0\|.$$

If $\lambda \in V$, then

$$|\langle x_0, \lambda \rangle| = |\lambda(x_0)| \leq \|\lambda\| \|x_0\| \leq \|x_0\|,$$

so

$$\sup_{\lambda \in V} |\langle x_0, \lambda \rangle| \leq \|x_0\|.$$

□

[3] Walter Rudin, *Functional Analysis*, second ed., p. 94, Theorem 4.3 (b).

[4] Walter Rudin, *Functional Analysis*, second ed., p. 58, Theorem 3.3.

Let X be a Banach space. For $x \in X$, it is apparent that $\lambda \mapsto \lambda(x)$ is a linear map $X^* \to \mathbb{C}$. From Theorem 4, it is bounded, with norm $\|x\|$. Define $\phi : X \to X^{**}$ by

$$(\phi x)(\lambda) = \langle x, \lambda \rangle, \qquad x \in X, \lambda \in X^*.$$

It is apparent that ϕ is a linear map. By Theorem 4, if $x \in X$ then $\|\phi x\| = \|x\|$, so ϕ is an isometry. Let $\phi x_n \in \phi(X)$ be a Cauchy sequence. $\phi^{-1} : \phi(X) \to X$ is an isometry, so $\phi^{-1}\phi x_n$ is a Cauchy sequence, i.e. x_n is a Cauchy sequence, and so, as X is a Banach space, x_n converges to some x. Then ϕx_n converges to ϕx, and thus $\phi(X)$ is a complete metric space. But a subset of a complete metric space is closed if and only if it is complete, so $\phi(X)$ is a closed subspace of X^{**}. Hence, $\phi(X)$ is a Banach space and $\phi : X \to \phi(X)$ is an isometric isomorphism. A Banach space is said to be **reflexive** if $\phi(X) = X^{**}$, i.e. if every bounded linear map $X^* \to \mathbb{C}$ is of the form $\phi(x)$ for some $x \in X$.

5 Adjoints

If X and Y are normed spaces and $T \in \mathscr{B}(X, Y)$, define $T^* : Y^* \to X^*$ by $T^*\lambda = \lambda \circ T$; as $T^*\lambda$ is the composition of two bounded linear maps it is indeed a bounded linear map $X \to \mathbb{C}$. T^* is called the **adjoint** of T. It is straightforward to check that T^* is linear and that it satisfies, for $S = T^*$,

$$\langle Tx, \lambda \rangle = \langle x, S\lambda \rangle, \qquad x \in X, \lambda \in Y^*. \tag{1}$$

On the other hand, suppose that $S : Y^* \to X^*$ is a function that satisfies (1). Let $\lambda \in Y^*$, and let $x \in X$. Then

$$(S\lambda)(x) = \lambda(Tx) = (T^*\lambda)(x).$$

This is true for all x, so $S\lambda = T^*\lambda$, and that is true for all λ, so $S = T^*$. Thus $T^*\lambda = \lambda \circ T$ is the unique function $Y^* \to X^*$ that satisfies (1), not just the unique bounded linear map that does. (That is, satisfying (1) completely determines a function.)

Using Theorem 4,

$$\begin{aligned}
\|T\| &= \sup_{\|x\| \le 1} \|Tx\| \\
&= \sup_{\|x\| \le 1} \sup_{\|\lambda\| \le 1} |\langle Tx, \lambda \rangle| \\
&= \sup_{\|x\| \le 1} \sup_{\|\lambda\| \le 1} |\langle x, T^*\lambda \rangle| \\
&= \sup_{\|\lambda\| \le 1} \sup_{\|x\| \le 1} |T^*\lambda(x)| \\
&= \sup_{\|\lambda\| \le 1} \|T^*\lambda\| \\
&= \|T^*\|.
\end{aligned}$$

In particular, $T^* \in \mathscr{B}(Y^*, X^*)$.

In the following we prove that the adjoint T^* of a compact operator T is itself a compact operator, and that if the adjoint of a bounded linear operator is compact then the original operator is compact.[5] In the proof we only show that if we take any sequence λ_n in the closed unit ball then it has a subsequence such that $T\lambda_{a(n)}$ converges. Check that it suffices merely to do this rather than showing that this happens for any bounded sequence.

Theorem 5. *If X and Y are Banach spaces and $T \in \mathscr{B}(X, Y)$, then T is compact if and only if T^* is compact.*

Proof. Suppose that $T \in \mathscr{B}(X, Y)$ is compact, and let $\lambda_n \in Y^*$, $n \geq 1$, be a sequence in the closed unit ball in Y^*.

If M is a metric space with metric ρ and $\mathcal{F}$ is a set of functions $M \to \mathbb{C}$, we say that $\mathcal{F}$ is **equicontinuous** if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $f \in \mathcal{F}$ and $\rho(x, y) < \delta$ then $|f(x) - f(y)| < \epsilon$. We say that $\mathcal{F}$ is **pointwise bounded** if for every $x \in M$ there is some $m(x) < \infty$ such that if $f \in \mathcal{F}$ and $x \in M$ then $|f(x)| \leq m(x)$. The **Arzelà-Ascoli theorem**[6] states that if (M, ρ) is a separable metric space and $\mathcal{F}$ is a set of functions $M \to \mathbb{C}$ that is equicontinuous and pointwise bounded, then for every sequence $f_n \in \mathcal{F}$ there is a subsequence that converges uniformly on every compact subset of M.

Let V be the closed unit ball in X. As T is a compact operator, $\overline{T(V)}$ is compact and therefore separable, because any compact metric space is separable. Define $f_n : \overline{T(V)} \to \mathbb{C}$ by

$$f_n(y) = \langle y, \lambda_n \rangle = \lambda_n(y).$$

For $y_1, y_2 \in \overline{T(V)}$ we have

$$|f_n(y_1) - f_n(y_2)| = |\lambda_n(y_1 - y_2)| \leq \|\lambda_n\| \|y_1 - y_2\| \leq \|y_1 - y_2\|.$$

Hence for $\epsilon > 0$, if $n \geq 1$ and $\|y_1 - y_2\| < \epsilon$ then $|f_n(y_1) - f_n(y_2)| < \epsilon$. This shows that $\{f_n\}$ is equicontinuous. If $y \in \overline{T(V)}$, then, for any $n \geq 1$,

$$|f_n(y)| = |\lambda_n(y)| \leq \|\lambda_n\| \|y\| \leq \|y\|,$$

showing that $\{f_n\}$ is pointwise bounded. Therefore we can apply the Arzelà-Ascoli theorem: there is a subsequence $f_{a(n)}$ such that $f_{a(n)}$ converges uniformly on every compact subset of $\overline{T(V)}$, in particular on $\overline{T(V)}$ itself and therefore on any subset of it, in particular $T(V)$. We are done using the Arzelà-Ascoli theorem: we used it to prove that there is a subsequence $f_{a(n)}$ that converges uniformly on $T(V)$.

Let $\epsilon > 0$. As $f_{a(n)}$ converges uniformly on $T(V)$, there is some N such that

[5] Walter Rudin, *Functional Analysis*, second ed., p. 105, Theorem 4.19.

[6] Walter Rudin, *Real and Complex Analysis*, third ed., p. 245, Theorem 11.28.

if $n, m \geq N$ and $y \in T(V)$, then $|f_{a(n)}(y) - f_{a(m)}(y)| < \epsilon$. Thus, if $n, m \geq N$,

$$\begin{aligned}
\|T^*\lambda_{a(n)} - T^*\lambda_{a(m)}\| &= \|\lambda_{a(n)} \circ T - \lambda_{a(m)} \circ T\| \\
&= \sup_{x \in V} |\lambda_{a(n)}(Tx) - \lambda_{a(m)}(Tx)| \\
&= \sup_{x \in V} |\lambda_{a(n)}(Tx) - \lambda_{a(m)}(Tx)| \\
&= \sup_{x \in V} |f_{a(n)}(Tx) - f_{a(m)}(Tx)| \\
&< \epsilon.
\end{aligned}$$

This means that $T^*\lambda_{a(n)} \in X^*$ is a Cauchy sequence. As X^* is a Banach space, this sequence converges, and therefore T^* is a compact operator.

Suppose that $T^* \in \mathscr{B}(Y^*, X^*)$ is compact. Therefore, by what we showed in the first half of the proof we have that $T^{**} : X^{**} \to Y^{**}$ is compact. If V be the closed unit ball in X^{**}, then $T^{**}(V)$ is totally bounded.

We have seen that $\phi : X \to X^{**}$ defined by $(\phi x)\lambda = \lambda(x)$, $x \in X$, $\lambda \in X^*$, is an isometric isomorphism $X \to \phi(X)$. Let $\psi : Y \to Y^{**}$ be the same for Y, and let U be the closed unit ball in X. If $x \in X$ and $\lambda \in Y^*$ then

$$\langle \lambda, \psi Tx \rangle = \langle Tx, \lambda \rangle = \langle x, T^*\lambda \rangle = \langle T^*\lambda, \phi x \rangle = \langle \lambda, T^{**}\phi x \rangle.$$

Therefore $\psi T = T^{**}\phi$. If $x \in U$ then $\phi x \in V$, as ϕ is an isometry. Hence if $x \in U$ then $\psi Tx = T^{**}\phi x \in T^{**}(V)$, thus

$$\psi T(U) \subseteq T^{**}(V).$$

As $\psi T(U)$ is contained in a totally bounded set it is itself totally bounded, and as ψ is an isometry, it follows that $T(U)$ is totally bounded. Hence T is a compact operator. □

6 Complemented subspaces

If M is a closed subspace of a topological vector space X and there exists a closed subspace N of X such that

$$X = M + N, \qquad M \cap N = \{0\},$$

we say that M **is complemented in** X and that X is the **direct sum** of M and N, which we write as $X = M \oplus N$.

We are going to use the following lemma to prove the theorem that comes after it.[7]

Lemma 6. *If X is a locally convex topological vector space and M is a subspace of X with $\dim X < \infty$, then M is complemented in X.*

[7] Walter Rudin, *Functional Analysis*, second ed., p. 106, Lemma 4.21.

In particular, a normed space is locally convex so the lemma applies to normed spaces. In the following theorem we prove that if $T \in \mathscr{B}(X)$ is compact and $\lambda \neq 0$ then $T - \lambda \mathrm{id}_X$ has closed image.[8]

Theorem 7. *If X is a Banach space, $T \in \mathscr{B}(X)$ is compact, and $\lambda \neq 0$, then the image of $T - \lambda \mathrm{id}_X$ is closed.*

Proof. According to Theorem 3, $\dim \ker(T - \lambda \mathrm{id}_X) < \infty$, and we can then use Lemma 6: $\ker(T - \lambda \mathrm{id}_X)$ is a finite dimensional subspace of the locally convex space X, so there is a closed subspace N of X such that $X = \ker(T - \lambda \mathrm{id}_X) \oplus N$.

Define $S : N \to X$ by $Sx = Tx - \lambda x$, so $S \in \mathscr{B}(N, X)$. It is apparent that $T(X) = S(N)$ and that S is injective, and we shall prove that $S(N)$ is closed. To show that $S(N)$ is closed, check that it suffices to prove that S is **bounded below**: that there is some $r > 0$ such that if $x \in N$ then $\|Sx\| \geq r\|x\|$.[9]

Suppose by contradiction that for every $r > 0$ there is some $x \in N$ such that $\|Sx\| < r\|x\|$. So for each $n \geq 1$, let $x_n \in N$ with $\|Sx_n\| < \frac{1}{n}\|x_n\|$, and put $v_n = \frac{x_n}{\|x_n\|}$, so that $\|v_n\| = 1$ and $\|Sv_n\| < \frac{1}{n}$. As T is compact, there is some subsequence such that $Tv_{a(n)}$ converges, say to v. Combining this with $Sv_n \to 0$ we get $\lambda v_{a(n)} \to v$. On the one hand, $\|\lambda v_{a(n)}\| = |\lambda| \|v_{a(n)}\| = |\lambda|$, so $\|v\| = |\lambda|$. On the other hand, since $\lambda v_{a(n)} \in N$ and N is closed, we get $v \in N$. S is continuous and $\lambda v_{a(n)} \to 0$, so

$$Sv = \lim_{n \to \infty} S(\lambda v_{a(n)}) = \lambda \lim_{n \to \infty} Sv_{a(n)} = 0.$$

Because S is injective and $Sv = 0$, we get $v = 0$, contradicting $\|v\| = |\lambda| > 0$. Therefore S is bounded below, and hence has closed image, completing the proof. □

The following theorem states that the point spectrum of a compact operator is countable and bounded, and that if there is a limit point of the point spectrum it is 0.[10] By countable we mean bijective with a subset of the integers.

Theorem 8. *If X is a Banach space, $T \in \mathscr{B}(X)$ is compact, and $r > 0$, then there are only finitely many eigenvalues λ of T such that $|\lambda| > r$.*

The following theorem shows that if $T \in \mathscr{B}(X)$ is compact and $\lambda \neq 0$, then the operator $T - \lambda \mathrm{id}_X$ is injective if and only if it is surjective.[11] This tells us that if $\lambda \neq 0$ is not an eigenvalue of T, then $T - \lambda \mathrm{id}_X$ is both injective and surjective, and hence is invertible, which means that if $\lambda \neq 0$ is not an eigenvalue of T then $\lambda \notin \sigma(T)$. This is an instance of the **Fredholm alternative**.

Theorem 9 (Fredholm alternative)**.** *Let X be a Banach space, $T \in \mathscr{B}(X)$ be compact, and $\lambda \neq 0$. $T - \lambda \mathrm{id}_X$ is injective if and only if it is surjective.*

[8] Walter Rudin, *Functional Analysis*, second ed., p. 107, Theorem 4.23.

[9] A common way of proving that a linear operator is invertible is by proving that it has dense image and that it is bounded below: bounded below implies injective and bounded below and dense image imply surjective.

[10] Walter Rudin, *Functional Analysis*, second ed., p. 107, Theorem 4.24.

[11] Paul Garrett, *Compact operators on Banach spaces: Fredholm-Riesz*, `http://www.math.umn.edu/~garrett/m/fun/fredholm-riesz.pdf`

Proof. Suppose that $T - \lambda \mathrm{id}_X$ is injective and let $V_n = (T - \lambda \mathrm{id}_X)^n X$, $n \geq 1$. If $(T - \lambda \mathrm{id}_X)^n x \in V_n$, then, as $(T - \lambda \mathrm{id}_X)x \in X$, we have

$$(T - \lambda \mathrm{id}_X)^{n-1}(T - \lambda \mathrm{id}_X)x \in V_{n-1},$$

so $V_n \supseteq V_{n-1}$. Thus

$$V_1 \supseteq V_2 \supseteq \cdots$$

Certainly V_n is a normed vector space. Define $T_n \in \mathscr{B}(V_n)$ by $T_n x = Tx$, namely, T_n is the restriction of T to V_n.

As T is a compact operator, by Theorem 7 we get that $V_1 = (T - \lambda \mathrm{id}_X)(X)$ is closed. Hence V_1 is a Banach space, being a closed subspace of a Banach space. Assume as induction hypothesis that V_n is a closed subset of X. Thus V_n is a Banach space, and $T_n \in \mathscr{B}(V_n)$ is a compact operator, as it is the restriction of the compact operator T to V_n. Therefore by Theorem 7, the image of $T_n - \lambda \mathrm{id}_X$ is closed, but this image is precisely V_{n+1}. Therefore, if $n \geq 1$ then V_n is a closed subspace of X.

Suppose by contradiction that there is some $x \notin (T - \lambda \mathrm{id}_X)X = V_1$. If $y \in X$ then

$$(T - \lambda \mathrm{id}_X)^n x - (T - \lambda \mathrm{id}_X)^{n+1} y = (T - \lambda \mathrm{id}_X)^n (x - (T - \lambda \mathrm{id}_X)y).$$

As $x \notin (T - \lambda \mathrm{id}_X)X$, we have $x - (T - \lambda \mathrm{id}_X)y \neq 0$. As we have supposed that $T - \lambda \mathrm{id}_X$ is injective, any positive power of it is injective, and hence the right hand side of the above equation is not 0. Thus $(T - \lambda \mathrm{id}_X)^n x \neq (T - \lambda \mathrm{id}_X)^{n+1} y$, and as $y \in X$ was arbitrary,

$$(T - \lambda \mathrm{id}_X)^n x \notin (T - \lambda \mathrm{id}_X)^{n+1} X.$$

However, of course $(T - \lambda \mathrm{id}_X)^n x \in V_n$, so if $n \geq 1$ then V_n strictly contains V_{n+1}.

Riesz's lemma states that if M is a normed space, N is a proper closed subspace of M, and $0 < r < 1$, then there is some $x \in M$ with $\|x\| = 1$ and $\inf_{y \in X} \|x - y\| \geq r$.[12] For each $n \geq 1$, using Riesz's lemma there is some $v_n \in V_n$, $\|v_n\| = 1$, such that

$$\inf_{y \in V_{n+1}} \|v_n - y\| \geq \frac{1}{2};$$

we proved that each V_n is closed and that V_n is a strictly decreasing sequence to allow us to use Riesz's lemma.

If $n, m \geq 1$, then $(T - \lambda \mathrm{id}_X)v_m \in V_{m+1}$ and check that $Tv_{m+n} \subseteq V_{m+n}$, so

$$Tv_m - Tv_{m+n} = \lambda v_m + (T - \lambda \mathrm{id}_X)v_m - Tv_{m+n} \in \lambda v_m + V_{m+1}.$$

[12] Paul Garrett, *Riesz's lemma*, `http://www.math.umn.edu/~garrett/m/fun/riesz_lemma.pdf` In this reference, Riesz's lemma is stated for Banach spaces, but the proof in fact works for normed spaces with no modifications.

From this and the definition of the sequence v_m, we get

$$\|Tv_m - Tv_{m+n}\| \geq |\lambda| \cdot \frac{1}{2}.$$

That is, the distance between any two terms in Tv_m is $\geq \frac{|\lambda|}{2}$, which is a fixed positive constant, hence Tv_m has no convergent subsequence. But $\|v_m\| = 1$, so v_m is bounded and therefore, as T is compact, the sequence Tv_m has a convergent subsequence, a contradiction. Therefore $T - \lambda \mathrm{id}_X$ is surjective.

Suppose that $T - \lambda \mathrm{id}_X$ is surjective. One checks that if a bounded linear operator is surjective then its adjoint is injective. For $x \in X$ and $\mu \in X^*$, $\langle \lambda \mathrm{id}_X x, \mu \rangle = \mu(\lambda x) = \lambda \mu(x) = \langle x, \lambda \mathrm{id}_{X^*} \mu \rangle$, so $(\lambda \mathrm{id}_X)^* = \lambda \mathrm{id}_{X^*}$. Hence $(T - \lambda \mathrm{id}_X)^* = T^* - \lambda \mathrm{id}_{X^*}$. T is compact so T^* is compact. As $T^* - \lambda \mathrm{id}_{X^*}$ is injective and T^* is compact, $T^* - \lambda \mathrm{id}_{X^*}$ is surjective, whence its adjoint $T^{**} - \lambda \mathrm{id}_{X^{**}} : X^{**} \to X^{**}$ is injective. One checks that if $S \in \mathscr{B}(X)$ and $S^{**} : X^{**} \to X^{**}$ is injective then S is injective; this is proved using the fact that $\phi : X \to X^{**}$ defined by $(\phi x)(\lambda) = \lambda(x)$ is an isometric isomorphism $X \to \phi(X)$. Using this, $T - \lambda \mathrm{id}_X$ is injective, completing the proof. □

7 Compact metric spaces

In the proof of Theorem 5 we stated the Arzelà-Ascoli theorem. First we state definitions again. If M is a metric space with metric ρ and $\mathcal{F}$ is a set of functions $M \to \mathbb{C}$, we say that $\mathcal{F}$ is **equicontinuous** if for all $\epsilon > 0$ there is some $\delta > 0$ such that $f \in \mathcal{F}$ and $\rho(x, y) < \delta$ imply that $|f(x) - f(y)| < \epsilon$. We say that $\mathcal{F}$ is **pointwise bounded** if for all $x \in M$ there is some $m(x)$ such that if $f \in \mathcal{F}$ then $|f(x)| \leq m(x)$. The **Arzelà-Ascoli theorem** states that if M is a separable metric space and $\mathcal{F}$ is equicontinuous and pointwise bounded, then every sequence in $\mathcal{F}$ has a sequence that converges uniformly on every compact subset of M.[13]

We are going to use a converse of the Arzelà-Ascoli theorem in the case of a compact metric space.[14] Let M be a compact metric space and let $C(M)$ be the set of continuous functions $M \to \mathbb{C}$. It does not take long to prove that with the norm $\|f\| = \sup_{x \in M} |f(x)|$, $C(M)$ is a Banach space.

Theorem 10. *Let (M, ρ) be a compact metric space and let $\mathcal{F} \subseteq C(M)$. $\mathcal{F}$ is precompact in $C(M)$ if and only if $\mathcal{F}$ is bounded and equicontinuous.*

Proof. Suppose that $\mathcal{F}$ is bounded and equicontinuous. To say that $\mathcal{F}$ is bounded is to say that there is some C such that if $f \in \mathcal{F}$ then $\|f\| \leq C$, and this implies that $\mathcal{F}$ is pointwise bounded. As M is compact it is separable, so the Arzelà-Ascoli theorem tells us that every sequence in $\mathcal{F}$ has a subsequence that converges on every compact subset of M. To say that a sequence of functions $M \to \mathbb{C}$ converges uniformly on the compact subsets of M is to say

[13] Walter Rudin, *Real and Complex Analysis*, third ed., p. 245, Theorem 11.28.
[14] John B. Conway, *A Course in Functional Analysis*, second ed., p. 175, Theorem 3.8.

that the sequence converges in the norm of the Banach space $C(M)$, and thus if $\mathcal{F}$ is a subset of $C(M)$, then to say that every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of M is to say that $\mathcal{F}$ is precompact in $C(M)$.

In the other direction, suppose that $\mathcal{F}$ is precompact. Hence it is totally bounded in $C(M)$. It is straightforward to verify that $\mathcal{F}$ is bounded. We have to show that $\mathcal{F}$ is equicontinuous. Let $\epsilon > 0$. As $\mathcal{F}$ is totally bounded, there are $f_1, \dots, f_n \in \mathcal{F}$ such that $\mathcal{F} \subseteq \bigcup_{k=1}^n B_{\epsilon/3}(f_k)$. As each $f_k : M \to \mathbb{C}$ is continuous and M is compact, there is some $\delta_k > 0$ such that if $\rho(x, y) < \delta_k$ then $|f_k(x) - f_k(y)| < \frac{\epsilon}{3}$. Let $\delta = \min_{1 \leq k \leq n} \delta_k$. If $f \in \mathcal{F}$ and $\rho(x, y) < \delta$, then, taking k such that $\|f - f_k\| < \frac{\epsilon}{3}$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y) \\ &< \|f - f_k\| + \frac{\epsilon}{3} + \|f_k - f\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon, \end{aligned}$$

showing that $\mathcal{F}$ is equicontinuous. □

We now show that if M is a compact metric space then the Banach space $C(M)$ has the **approximation property**: every compact linear operator $C(M) \to C(M)$ is the limit of a sequence of bounded finite rank operators.[15]

Theorem 11. *If (M, ρ) is a compact metric space, then $\mathscr{B}_{00}(C(M))$ is a dense subset of $\mathscr{B}_0(C(M))$.*

Proof. Let $T \in \mathscr{B}_0(C(M))$, let V be the closed unit ball in $C(M)$, and let $\epsilon > 0$. Because $T(V)$ is precompact in $C(M)$, by Theorem 10 it is bounded and equicontinuous. Then there is some $\delta > 0$ such that if $Tf \in T(V)$ and $\rho(x, y) < \delta$ then $|(Tf)(x) - (Tf)(y)| < \epsilon$. M is compact, so there are $x_1, \dots, x_n \in M$ such that $M = \bigcup_{j=1}^n B_\delta(x_j)$. It is a fact that there is a **partition of unity** that is **subordinate** to this open covering of M: there are continuous functions $\phi_1, \dots, \phi_n : M \to [0, 1]$ such that if $x \in M$ then $\sum_{j=1}^n \phi_j(x) = 1$, and $\phi_j(x) = 0$ if $x \notin B_\delta(x_j)$.[16] Define $T_\epsilon : C(M) \to C(M)$ by

$$T_\epsilon f = \sum_{j=1}^n (Tf)(x_j)\phi_j.$$

It is apparent that T_ϵ is linear. $\|T_\epsilon f\| \leq \sum_{j=1}^n \|T\|\|f\| = n\|T\|\|f\|$, so $\|T_e\| \leq n\|T\|$. And the image of T_ϵ is contained in the span of $\{\phi_1, \dots, \phi_n\}$. Therefore $T_\epsilon \in \mathscr{B}_{00}(C(M))$.

[15] John B. Conway, *A Course in Functional Analysis*, second ed., p. 176, Theorem 3.11.
[16] John B. Conway, *Functional Analysis*, second ed., p. 139, Theorem 6.5.

If $f \in V$ and $x \in M$, then for each j either $x \in B_\delta(x_j)$, in which case $|(Tf)(x) - (Tf)(x_j)| < \epsilon$, or $x \notin B_\delta(x_j)$, in which case $\phi_j(x) = 0$. This gives us

$$\begin{aligned} |((Tf)(x) - (T_\epsilon f)(x)| &= \left| (Tf)(x) \cdot \sum_{j=1}^{n} \phi_j(x) - \sum_{j=1}^{n} (Tf)(x_j)\phi_j(x) \right| \\ &= \left| \sum_{j=1}^{n} \left((Tf)(x) - (Tf)(x_j)\right) \phi_j(x) \right| \\ &\leq \sum_{j=1}^{n} |(Tf)(x) - (Tf)(x_j)| \, \phi_j(x) \\ &< \sum_{j=1}^{n} \epsilon \phi_j(x) \\ &= \epsilon, \end{aligned}$$

showing that $\|Tf - T_\epsilon f\| < \epsilon$, and as this is true for all $f \in V$ we get $\|T - T_\epsilon\| < \epsilon$. □