Diophantine numbers

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1 Baire category

For real $\tau, \gamma > 0$, let $\mathcal{D}(\tau, \gamma)$ be the set of those $\xi \in \mathbb{R}$ such that for all $q \in \mathbb{Z}_{\geq 1}$ and $p \in \mathbb{Z}$,

$$\left| \xi - \frac{p}{q} \right| \geq \gamma q^{-\tau}.$$

In other words,

$$\mathcal{D}(\tau, \gamma) = \bigcap_{q \in \mathbb{Z}_{\geq 1}, p \in \mathbb{Z}} \left\{ \xi \in \mathbb{R} : \left| \xi - \frac{p}{q} \right| \geq \gamma q^{-\tau} \right\} = \bigcap_{q \in \mathbb{Z}_{\geq 1}, p \in \mathbb{Z}} \mathcal{D}(\tau, \gamma, q, p).$$

Each set $\mathcal{D}(\tau, \gamma, q, p)$ is closed, so $\mathcal{D}(\tau, \gamma)$ is closed. Let

$$\mathcal{D}(\tau) = \bigcup_{\gamma > 0} \mathcal{D}(\tau, \gamma).$$

Now, if $\gamma_1 \geq \gamma_2$ and $\xi \in \mathcal{D}(\tau, \gamma_1)$, take $q \in \mathbb{Z}_{\geq 1}$ and $p \in \mathbb{Z}$, and then

$$\left| \xi - \frac{p}{q} \right| \geq \gamma_1 q^{-\tau} \geq \gamma_2 q^{-\tau}.$$

Hence

$$\mathcal{D}(\tau, \gamma_1) \subset \mathcal{D}(\tau, \gamma_2), \qquad \gamma_1 \geq \gamma_2,$$

and thus $\mathcal{D}(\tau) = \bigcup_{n \geq 1} \mathcal{D}(\tau, n^{-1})$, which means that $\mathcal{D}(\tau)$ is an F_σ set. Let

$$\mathcal{D} = \bigcup_{\tau > 0} \mathcal{D}(\tau) = \bigcup_{\tau > 0} \bigcup_{\gamma > 0} \bigcap_{q \in \mathbb{Z}_{\geq 1}, p \in \mathbb{Z}} \mathcal{D}(\tau, \gamma, q, p),$$

whose elements are called **Diophantine numbers**. If $\tau_1 \leq \tau_2$ and $\xi \in \mathcal{D}(\tau_1)$, then for some $\gamma > 0$, $\xi \in \mathcal{D}(\tau_1, \gamma)$. Take $q \in \mathbb{Z}_{\geq 1}$ and $p \in \mathbb{Z}$. Then

$$\left| \xi - \frac{p}{q} \right| \geq \gamma q^{-\tau_1} \geq \gamma q^{-\tau_2},$$

so $\xi \in \mathcal{D}(\tau_2, \gamma)$, i.e.

$$\mathcal{D}(\tau_1) \subset \mathcal{D}(\tau_2), \qquad \tau_1 \leq \tau_2.$$

Thus $\mathcal{D} = \bigcup_{m \geq 1} \mathcal{D}(m) = \lim_{m \to \infty} \mathcal{D}(m)$, which means that $\mathcal{D}$ is an F_σ set.

Now,

$$\mathcal{D}^c = \bigcap_{m \geq 1} \bigcap_{n \geq 1} \bigcup_{q \in \mathbb{Z}_{\geq 1}, p \in \mathbb{Z}} \mathcal{D}(m, n^{-1}, q, p)^c,$$

and

$$\mathcal{D}(m, n^{-1}, q, p)^c = \left\{ \xi \in \mathbb{R} : \left| \xi - \frac{p}{q} \right| < n^{-1} q^{-m} \right\}.$$

For $\xi = \frac{p}{q} \in \mathbb{Q}$ with $q \geq 1$, $\xi \in \mathcal{D}(m, n^{-1}, q, p)^c$, so for each $m, n \geq 1$,

$$\mathbb{Q} \subset \bigcup_{q \in \mathbb{Z}_{\geq 1}, p \in \mathbb{Z}} \mathcal{D}(m, n^{-1}, q, p)^c.$$

This means that each set $\bigcup_{q \in \mathbb{Z}_{\geq 1}, p \in \mathbb{Z}} \mathcal{D}(m, n^{-1}, q, p)^c$ is dense. It is a union of open sets hence open, so it is an open dense set. Therefore $\mathcal{D}^c$ is a comeager set and $\mathcal{D}$ is a meager set. Because $\mathbb{R}$ is a complete metric space, by the Baire category theorem $\mathcal{D}^c$ is dense.

2 Algebraic numbers

Suppose that $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and that $f(\xi) = 0$ for a nonzero irreducible polynomial $f \in \mathbb{Z}[x]$ of degree d. Let

$$M = \sup\{|f'(x)| : x \in [\xi - 1, \xi + 1]\}, \qquad A = \min(1, M^{-1}),$$

and suppose by contradiction that there are $q \in \mathbb{Z}_{\geq 1}$ and $p \in \mathbb{Z}$ such that

$$\left| \xi - \frac{p}{q} \right| < A q^{-d}.$$

By the mean-value theorem there is some c between ξ and $\frac{p}{q}$ such that

$$f(\xi) - f(p/q) = f'(c) \left(\xi - \frac{p}{q} \right),$$

and as $f(\xi) = 0$,

$$|f'(c)| \left| \xi - \frac{p}{q} \right| = |f(p/q)|.$$

As f is irreducible, $f(p/q) \neq 0$. But then $f(p/q)$ is a rational number which in lowest terms has denominator $\leq q^d$, therefore $|f(p/q)| \geq q^{-d}$, giving

$$\left| \xi - \frac{p}{q} \right| \geq \frac{1}{|f'(c)|} q^{-d}.$$

Furthermore, $\left|\xi - \frac{p}{q}\right| < A \leq 1$ so $\frac{p}{q} \in [\xi - 1, \xi + 1]$, and as c is between ξ and $\frac{p}{q}$ it holds that $c \in [\xi - 1, \xi + 1]$ and thus $|f'(c)| \leq M$. Therefore

$$\left|\xi - \frac{p}{q}\right| \geq M^{-1} q^{-d} \geq A q^{-d},$$

a contradiction. Therefore for all $q \in \mathbb{Z}_{\geq 1}$ and $p \in \mathbb{Z}$,

$$\left|\xi - \frac{p}{q}\right| \geq A q^{-d}.$$

This means that $\xi \in \mathcal{D}(d, A)$.

Theorem 1 (Liouville approximation theorem)**.** *If $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and $f(\xi) = 0$ for a nonzero irreducible polynomial $f \in \mathbb{Z}[x]$ of degree d, then $\xi \in \mathcal{D}(d)$.*

We remark for the above theorem that because ξ is irrational, $d \geq 2$. Also, the above theorem shows that the set of real irrational algebraic numbers is contained in $\mathcal{D}$, which implies that if $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and $\xi \in \mathcal{D}^c$ then ξ is transcendental. Elements of $(\mathbb{R} \setminus \mathbb{Q}) \cap \mathcal{D}^c$ are called **Liouville numbers**. **Roth's theorem** states that the set of real irrational algebraic numbers is contained in

$$\mathcal{D}(2+) = \bigcap_{\tau > 2} \mathcal{D}(\tau).$$

3 Measure theory

Take $\tau > 2$ and $\gamma > 0$.

$$\mathcal{D}(\tau, \gamma)^c = \bigcup_{q \in \mathbb{Z}_{\geq 1}} \bigcup_{p \in \mathbb{Z}} \left\{ \xi \in \mathbb{R} : \left|\xi - \frac{p}{q}\right| < \gamma q^{-\tau} \right\}.$$

For $m \in \mathbb{Z}_{\geq 1}$,

$$\mathcal{D}(\tau, \gamma)^c \cap (-m, m) = \bigcup_{q \in \mathbb{Z}_{\geq 1}} \bigcup_{|p| \leq mq + \gamma q^{-\tau+1}} \left\{ -m < \xi < m : \left|\xi - \frac{p}{q}\right| < \gamma q^{-\tau} \right\}.$$

Hence

$$\begin{aligned}
\lambda(\mathcal{D}(\tau, \gamma)^c \cap (-m, m)) &\leq \sum_{q \geq 1} (2mq + 2\gamma q^{-\tau+1} + 1) \cdot 2\gamma q^{-\tau} \\
&= \sum_{q \geq 1} 4m\gamma q^{-\tau+1} + \sum_{q \geq 1} 4\gamma^2 q^{-2\tau+1} + \sum_{q \geq 1} 2\gamma q^{-\tau} \\
&= 4m\gamma\zeta(\tau - 1) + 4\gamma^2\zeta(2\tau - 1) + 2\gamma\zeta(\tau).
\end{aligned}$$

Therefore for each $\gamma > 0$,

$$\lambda(\mathcal{D}(\tau)^c \cap (-m, m)) \leq 4m\gamma\zeta(\tau - 1) + 4\gamma^2\zeta(2\tau - 1) + 2\gamma\zeta(\tau),$$

hence $\lambda(\mathcal{D}(\tau)^c \cap (-m, m)) = 0$. This is true for each $m \in \mathbb{Z}_{\geq 1}$, therefore $\lambda(\mathcal{D}(\tau)^c) = 0$.

4 Exponentials

Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda = e^{2\pi i \xi}$. For $q \in \mathbb{Z}_{\geq 1}$ and $p_q \in \mathbb{Z}$ closest to $q\xi$,

$$|\lambda^q - 1| = |e^{2\pi i q\xi} - e^{2\pi i p_q}| \leq |2\pi q\xi - 2\pi p_q| = 2\pi|q\xi - p_q|,$$

and, as $|q\xi - p_q| \leq \frac{1}{2}$ and since $|\sin x| \geq \frac{2}{\pi}|x|$ for $|x| \leq \frac{\pi}{2}$,

$$\begin{aligned}
|\lambda^q - 1| &= |e^{2\pi i(q\xi - p_q)} - 1| \\
&= |e^{\pi i(q\xi - p_q)} - e^{-\pi i(q\xi - p_q)}| \\
&= 2|\sin \pi(q\xi - p_q)| \\
&\geq 2 \cdot \frac{2}{\pi}|\pi(q\xi - p_q)| \\
&= 4|q\xi - p_q|.
\end{aligned}$$

Thus

$$4|q\xi - p_q| \leq |\lambda^q - 1| \leq 2\pi|q\xi - p_q|.$$

If $\xi \in \mathcal{D}(\tau, \gamma)$, then, for $q \in \mathbb{Z}_{\geq 1}$,

$$|\lambda^q - 1|^{-1} \leq \frac{1}{4|q\xi - p_q|} = \frac{1}{4q}\left|\xi - \frac{p_q}{q}\right|^{-1} \leq \frac{1}{4q}\gamma^{-1}q^\tau = \frac{\gamma^{-1}}{4}q^{\tau-1}.$$

Siegel's linearization theorem states that for $\xi \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda = e^{2\pi i\xi}$, if $|\lambda^q - 1|$ is majorized by a power of q, then if f is holomorphic on a neighborhood of 0 and $f'(0) = \lambda$ then f is linearizable at 0.[1] Thus if $\xi \in \mathcal{D}$ and $f'(0) = \lambda = e^{2\pi i\xi}$, then f is linearizable at 0.

[1] John Milnor, *Dynamics in One Complex Variable*, third ed., p. 127, Theorem 11.4.