

# Gaussian integrals

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## 1 One dimensional Gaussian integrals

For  $p \in \mathbb{C}$ , let<sup>1</sup>

$$h(p) = \int_{\mathbb{R}} e^{-x^2/2} e^{-ipx} dx.$$

Then we check that

$$h'(p) = -i \int_{\mathbb{R}} x e^{-x^2/2} e^{-ipx} dx = i \int_{\mathbb{R}} \frac{d}{dx} (e^{-x^2/2}) e^{-ipx} dx.$$

Integrating by parts yields

$$h'(p) = -p \int_{\mathbb{R}} e^{-x^2/2} e^{-ipx} dx = -ph(p).$$

Since  $h'(p) = -ph(p)$ ,<sup>2</sup>

$$h(p) = h(0)e^{-p^2/2}.$$

Now, using Fubini's theorem and then polar coordinates,

$$\begin{aligned} h(0)^2 &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-x^2/2} dx \right) e^{-y^2/2} dy \\ &= \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2} dx dy \\ &= \int_0^\infty \left( \int_{S^1} e^{-r^2/2} e^{-r^2/2} d\sigma(\theta) \right) r dr \\ &= 2\pi \int_0^\infty r e^{-r^2/2} dr \\ &= 2\pi, \end{aligned}$$

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<sup>1</sup>Eberhard Zeidler, *Quantum Field Theory I: Basics in Mathematics and Physics*, p. 493, Problem 7.1.

<sup>2</sup>cf. Einar Hille, *Ordinary Differential Equations in the Complex Domain*.

so

$$h(p) = (2\pi)^{1/2} e^{-p^2/2}.$$

For  $a > 0$  and  $p \in \mathbb{C}$ , doing the change of variable  $y = a^{1/2}x$ ,

$$\begin{aligned} (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ax^2/2} e^{-ipx} dx &= (2\pi)^{-1/2} a^{-1/2} \int_{\mathbb{R}} e^{-y^2/2} e^{-ipa^{-1/2}y} dy \\ &= (2\pi)^{-1/2} a^{-1/2} h(pa^{-1/2}) \\ &= a^{-1/2} e^{-a^{-1}p^2/2}. \end{aligned}$$

For  $t > 0$  and  $m \in \mathbb{R}$ , doing the change of variable  $y = x - m$ , and using the above with  $a = t^{-2}$  and  $p = 0$ ,

$$\begin{aligned} (2\pi t^2)^{-1/2} \int_{\mathbb{R}} \exp\left(-\frac{(x-m)^2}{2t^2}\right) dx &= (2\pi)^{-1/2} t^{-1} \int_{\mathbb{R}} e^{-ay^2/2} dx \\ &= t^{-1} \cdot a^{-1/2} \\ &= 1. \end{aligned}$$

**Theorem 1.** For  $a > 0$  and  $p \in \mathbb{C}$ ,

$$(2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ax^2/2} e^{-ipx} dx = a^{-1/2} e^{-a^{-1}p^2/2}.$$

For  $t > 0$  and  $m \in \mathbb{R}$ ,

$$(2\pi t^2)^{-1/2} \int_{\mathbb{R}} \exp\left(-\frac{(x-m)^2}{2t^2}\right) dx = 1.$$

For  $t > 0$  and  $x \in \mathbb{R}$ , let

$$p_t(x) = (2\pi t^2)^{-1/2} \exp\left(-\frac{x^2}{2t^2}\right).$$

For  $\phi \in \mathcal{S}(\mathbb{R})$ , doing the change of variable  $x = ty$ ,

$$\int_{\mathbb{R}} \phi(x) p_t(x) dx = (2\pi)^{-1/2} \int_{\mathbb{R}} \phi(ty) e^{-y^2/2} dy = \int_{\mathbb{R}} \phi(tx) p_1(x) dx.$$

Then as  $t \downarrow 0$ , using the dominated convergence theorem,

$$\int_{\mathbb{R}} \phi(x) p_t(x) dx \rightarrow \int_{\mathbb{R}} \phi(0) p_1(x) dx = \phi(0).$$

For  $\phi \in L^1(\mathbb{R}^N)$ , let

$$\widehat{\phi}(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-i\langle \xi, x \rangle} \phi(x) dx, \quad \xi \in \mathbb{R}^N.$$

By Theorem 1, with  $a = t^{-2}$ ,

$$\begin{aligned}\widehat{p}_t(\xi) &= (2\pi t^2)^{-1/2} \cdot (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ax^2/2} e^{-i\xi x} dx \\ &= (2\pi t^2)^{-1/2} a^{-1/2} e^{-a^{-1}\xi^2/2} \\ &= (2\pi)^{-1/2} e^{-t^2\xi^2/2}.\end{aligned}$$

## 2 Moments

For  $a > 0$ , define for  $Z \in \mathbb{C}$ ,

$$Z(J) = a^{1/2}(2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ax^2/2} e^{iJx} dx.$$

By Theorem 1,

$$Z(J) = e^{-a^{-1}J^2/2}. \quad (1)$$

By the dominated convergence theorem,

$$Z^{(n)}(J) = a^{1/2}(2\pi)^{-1/2} i^n \int_{\mathbb{R}} x^n e^{-ax^2/2} e^{iJx} dx,$$

and so

$$a^{1/2}(2\pi)^{-1/2} \int_{\mathbb{R}} x^n e^{-ax^2/2} dx = i^{-n} \frac{dZ}{dJ}(0).$$

From (1) we calculate

$$Z'(J) = -a^{-1}JZ(J), \quad Z''(J) = -a^{-1}Z(J) + a^{-2}J^2Z(J),$$

so  $Z''(0) = -a^{-1}Z(0) = -a^{-1}$ , and thus for  $t > 0$  and  $a = t^{-1}$ ,

$$(2\pi t)^{-1/2} \int_{\mathbb{R}} x^2 e^{-t^{-2}x^2/2} dx = t,$$

i.e.

$$\int_{\mathbb{R}} x^2 p_t(x) dx = t.$$

## 3 $N$ -dimensional Gaussian integrals

Let  $S(x) = \frac{\langle x, x \rangle}{2}$  for  $x \in \mathbb{R}^N$ . For  $\chi \in \mathcal{D}(\mathbb{R}^N)$  and  $t > 0$ , **Laplace's method**<sup>3</sup> tells us that

$$\int_{\mathbb{R}^N} e^{-tS(x)} \chi(x) dx = (2\pi t^{-1})^{N/2} (\det \text{Hess } S(0))^{-1/2} e^{-tS(0)} \chi(0) (1 + O(t^{-1}))$$

<sup>3</sup>cf. <http://individual.utoronto.ca/jordanbell/notes/stationaryphase.pdf>, p. 5, §5.

as  $t \rightarrow \infty$ . Here,  $\text{Hess } S(x) = I$  for all  $x$  and  $S(0) = 0$ , so

$$\int_{\mathbb{R}^N} e^{-t \frac{\langle x, x \rangle}{2}} \chi(x) dx = (2\pi t^{-1})^{N/2} \chi(0) (1 + O(t^{-1}))$$

as  $t \rightarrow \infty$ .

For  $A$  an  $N \times N$  matrix, we write  $A > 0$  if  $A$  is symmetric and has positive eigenvalues. It is proved that<sup>4</sup>

$$\begin{aligned} & \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2} \langle Ax, x \rangle - i \langle \xi, x \rangle\right) dx \\ &= (\det A)^{-1/2} (2\pi)^{N/2} \exp\left(-\frac{1}{2} \langle A^{-1} \xi, \xi \rangle\right) \end{aligned}$$

for all  $\xi \in \mathbb{R}^N$ , and

$$\int_{\mathbb{R}^N} \exp\left(-\frac{1}{2} \langle Ax, x \rangle + \langle b, x \rangle\right) dx = (\det A)^{-1/2} (2\pi)^{N/2} \exp\left(\frac{1}{2} \langle A^{-1} b, b \rangle\right).$$

for all  $b \in \mathbb{R}^N$ . Let

$$Z_A = \int_{\mathbb{R}^N} e^{-\frac{1}{2} \langle Ax, x \rangle} dx = (\det A)^{-1/2} (2\pi)^{N/2}.$$

Let  $\lambda_N$  be Lebesgue measure on  $\mathbb{R}^N$  and let  $\mu_A$  be the following Borel probability measure on  $\mathbb{R}^N$ :<sup>5</sup>

$$d\mu_A(x) = \frac{1}{Z_A} e^{-\frac{1}{2} \langle Ax, x \rangle} d\lambda_N(x) = (\det A)^{1/2} (2\pi)^{-N/2} e^{-\frac{1}{2} \langle Ax, x \rangle} d\lambda_N(x).$$

For  $\xi \in \mathbb{R}^N$ ,

$$\begin{aligned} \int_{\mathbb{R}^N} e^{-i \langle \xi, x \rangle} d\mu_A(x) &= (\det A)^{1/2} (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-\frac{1}{2} \langle Ax, x \rangle} e^{-i \langle \xi, x \rangle} d\lambda_N(x) \\ &= (\det A)^{1/2} (2\pi)^{-N/2} \cdot (\det A)^{-1/2} (2\pi)^{N/2} e^{-\frac{1}{2} \langle A^{-1} \xi, \xi \rangle} \\ &= e^{-\frac{1}{2} \langle A^{-1} \xi, \xi \rangle}, \end{aligned}$$

and for  $b \in \mathbb{R}^N$ ,

$$\begin{aligned} \int_{\mathbb{R}^N} e^{\langle b, x \rangle} d\mu_A(x) &= (\det A)^{1/2} (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{\langle b, x \rangle} e^{-\frac{1}{2} \langle Ax, x \rangle} d\lambda_N(x) \\ &= (\det A)^{1/2} (2\pi)^{-N/2} \cdot (\det A)^{-1/2} (2\pi)^{N/2} e^{\frac{1}{2} \langle A^{-1} b, b \rangle} \\ &= e^{\frac{1}{2} \langle A^{-1} b, b \rangle}. \end{aligned}$$

<sup>4</sup><http://individual.utoronto.ca/jordanbell/notes/stationaryphase.pdf>, Theorem 2 and Theorem 3.

<sup>5</sup>See <http://individual.utoronto.ca/jordanbell/notes/gaussian.pdf>, p. 6, §4.

**Theorem 2.** For  $\xi \in \mathbb{R}^N$ ,

$$\int_{\mathbb{R}^N} e^{-i\langle \xi, x \rangle} d\mu_A(x) = e^{-\frac{1}{2}\langle A^{-1}\xi, \xi \rangle},$$

and for  $b \in \mathbb{R}^N$ ,

$$\int_{\mathbb{R}^N} e^{\langle b, x \rangle} d\mu_A(x) = e^{\frac{1}{2}\langle A^{-1}b, b \rangle}.$$

Let<sup>6</sup>

$$L = L^A = \sum_{j,k=1}^N A_{j,k}^{-1} \partial_j \partial_k.$$

We work out the semigroup whose infinitesimal generator is  $L/2$ .<sup>7</sup>

**Theorem 3.** For  $f \in C^1(\mathbb{R}^N)$  that is  $\mu_A$ -integrable and for  $t > 0$ ,

$$(e^{tL/2}f)(x) = \int_{\mathbb{R}^N} f(x - t^{1/2}y) d\mu_A(y), \quad x \in \mathbb{R}^N.$$

*Proof.* For  $\xi \in \mathbb{R}^N$  define  $f(x) = e^{\langle \xi, x \rangle} = e^{\xi_1 x_1 + \dots + \xi_N x_N}$ . On the one hand,

$$Lf = \sum_{j,k=1}^n A_{j,k}^{-1} \xi_j \xi_k f = \langle A^{-1}\xi, \xi \rangle f.$$

Then

$$\exp(tL/2)f = \exp\left(\frac{1}{2}t \langle A^{-1}\xi, \xi \rangle\right) f.$$

On the other hand, for  $x \in \mathbb{R}^N$ , applying Theorem 2,

$$\begin{aligned} \int_{\mathbb{R}^N} f(x - t^{1/2}y) d\mu_A(y) &= \int_{\mathbb{R}^N} e^{\langle \xi, x - t^{1/2}y \rangle} d\mu_A(y) \\ &= e^{\langle \lambda, x \rangle} \int_{\mathbb{R}^N} e^{\langle -t^{1/2}\xi, y \rangle} d\mu_A(y) \\ &= e^{\langle \lambda, x \rangle} e^{\frac{1}{2}\langle A^{-1}(-t^{1/2}\xi), (-t^{1/2}\xi) \rangle} \\ &= e^{\frac{1}{2}t \langle A^{-1}\xi, \xi \rangle} f. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N} f(x - t^{1/2}y) d\mu_A(y) = e^{tL/2}f.$$

□

<sup>6</sup>See <http://www.math.ucsd.edu/~bdriver/247A-Winter2012/>

<sup>7</sup>cf. the Ornstein-Uhlenbeck semigroup, <http://individual.utoronto.ca/jordanbell/notes/gaussian.pdf>, §6.

## 4 Concentration of measure

Let  $\gamma_N$  be the Borel probability measure on  $\mathbb{R}^N$  defined by

$$d\gamma_N(x) = (2\pi)^{-N/2} e^{-\frac{1}{2}\langle x, x \rangle} d\lambda_N(x).$$

We estimate the mass  $\gamma_N$  assigns to a spherical shell about the sphere of radius  $N^{1/2}$ .<sup>8</sup>

**Theorem 4.** For  $\delta \geq 0$ ,

$$\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \geq N + \delta\} \leq \left(\frac{N}{N + \delta}\right)^{-N/2} e^{-\delta/2},$$

and for  $0 < \delta \leq N$ ,

$$\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \leq N - \delta\} \leq \left(\frac{N}{N - \delta}\right)^{-N/2} e^{\delta/2}.$$

*Proof.* For  $0 < \lambda < 1$ , if  $\|x\|^2 \geq N + \delta$  then  $\lambda \|x\|^2 / 2 \geq \lambda(N + \delta)/2$  and then  $e^{\lambda \|x\|^2 / 2} \geq e^{\lambda(N + \delta)/2}$ . Hence

$$\begin{aligned} \gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \geq N + \delta\} &= e^{-\lambda(N + \delta)/2} \int_{\|x\|^2 \geq N + \delta} e^{\lambda(N + \delta)/2} d\gamma_N(x) \\ &\leq e^{-\lambda(N + \delta)/2} \int_{\|x\|^2 \geq N + \delta} e^{\lambda \|x\|^2 / 2} d\gamma_N(x) \\ &\leq e^{-\lambda(N + \delta)/2} \int_{\mathbb{R}^N} e^{\lambda \|x\|^2 / 2} d\gamma_N(x) \\ &= e^{-\lambda(N + \delta)/2} \cdot (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{\lambda \|x\|^2 / 2} e^{-\frac{1}{2} \|x\|^2} d\lambda_N(x) \\ &= e^{-\lambda(N + \delta)/2} \cdot \prod_{k=1}^N (2\pi)^{-1/2} \int_{\mathbb{R}} e^{(\lambda - 1)u^2 / 2} du. \end{aligned}$$

For  $a = -\lambda + 1 > 0$ , we have by Theorem 1

$$(2\pi)^{-1/2} \int_{\mathbb{R}} e^{-au^2 / 2} du = a^{-1/2},$$

so

$$\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \geq N + \delta\} \leq e^{-\lambda(N + \delta)/2} a^{-N/2} = e^{-\lambda(N + \delta)/2} (1 - \lambda)^{-N/2}.$$

For  $\lambda = \frac{\delta}{N + \delta}$  this is

$$\gamma_N\{x \in \mathbb{R}^N : \|x\|^2 \geq N + \delta\} \leq e^{-\delta/2} \left(\frac{N}{N + \delta}\right)^{-N/2}.$$

□

<sup>8</sup>Alexander Barvinok, *Measure Concentration*, <http://www.math.lsa.umich.edu/~barvinok/total710.pdf>, p. 5, Proposition 2.2.

Let  $\Sigma_N = \{x \in \mathbb{R}^N : \|x\| = N^{1/2}\}$ , and let  $\mu_N$  be the unique  $SO(N)$ -invariant regular Borel probability measure on  $S^{N-1}$ .<sup>9</sup> Let  $\pi_N : \Sigma_N \rightarrow \mathbb{R}$  be the projection

$$\pi_N(x) = \pi_N(x_1, \dots, x_N) = x_1,$$

and let  $\nu_N = (\pi_N)_* \mu_N$ , the pushforward measure which is itself a Borel probability measure on  $\mathbb{R}$ . The following theorem states that the measures  $\nu_N$  converges strongly to the standard Gaussian measure  $\gamma_1$ .<sup>10</sup>

**Theorem 5.** *For  $A$  a Borel set in  $\mathbb{R}$ ,*

$$\nu_N(A) \rightarrow \gamma_1(A)$$

as  $N \rightarrow \infty$ .

## 5 Zeta functions

Let  $A > 0$ , with eigenvalues  $\lambda_1, \dots, \lambda_N$ , counted according to multiplicity. For  $s \in \mathbb{C}$ , define<sup>11</sup>

$$\zeta_A(s) = \sum_{k=1}^N \lambda_k^{-s} = \sum_{k=1}^N e^{-s \log \lambda_k}.$$

The derivative of  $\zeta_A$  is

$$\zeta'_A(s) = \sum_{k=1}^N -\log \lambda_k \cdot \lambda_k^{-s},$$

so

$$\zeta'_A(0) = -\sum_{k=1}^N \log \lambda_k,$$

hence

$$e^{-\zeta'_A(0)} = \prod_{k=1}^N \lambda_k = \det A.$$

**Theorem 6.** *For  $\xi \in \mathbb{R}^N$ ,*

$$(2\pi)^{-N/2} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2} \langle Ax, x \rangle - i \langle \xi, x \rangle\right) dx = e^{\zeta'_A(0)/2} \exp\left(-\frac{1}{2} \langle A^{-1} \xi, \xi \rangle\right).$$

<sup>9</sup><http://individual.utoronto.ca/jordanbell/notes/harmonicpolynomials.pdf>, §2.

<sup>10</sup>Alexander Barvinok, *Measure Concentration*, <http://www.math.lsa.umich.edu/~barvinok/total710.pdf>, p. 54, Theorem 13.2.

<sup>11</sup>Eberhard Zeidler, *Quantum Field Theory I: Basics in Mathematics and Physics*, p. 434, §7.23.3.

Let  $\lambda_k > 0$ ,  $k \geq 1$ , and let  $Ae_k = \lambda_k e_k$ , and if it makes sense let<sup>12</sup>

$$\det A = \prod_{k=1}^{\infty} \lambda_k.$$

For those complex  $s$  for which the expression makes sense, let<sup>13</sup>

$$\zeta_A(s) = \sum_{k=1}^{\infty} \lambda_k^{-s} = \sum_{k=1}^{\infty} e^{-s \log \lambda_k}.$$

Then, if the above makes sense in a neighborhood of  $s = 0$ ,

$$\zeta'_A(0) = - \sum_{k=1}^{\infty} \log \lambda_k,$$

so

$$e^{-\zeta'_A(0)} = \det A.$$

We calculate, doing the change of variables  $t = \lambda_k u$ ,

$$\begin{aligned} \Gamma(s)\zeta_A(s) &= \int_0^{\infty} t^{s-1} e^{-t} dt \cdot \sum_{k=1}^{\infty} \lambda_k^{-s} \\ &= \sum_{k=1}^{\infty} \lambda_k^{-s} \int_0^{\infty} t^{s-1} e^{-t} dt \\ &= \sum_{k=1}^{\infty} \lambda_k^{-s} \int_0^{\infty} (\lambda_k u)^{s-1} e^{-\lambda_k u} \lambda_k du \\ &= \sum_{k=1}^{\infty} \int_0^{\infty} u^{s-1} e^{-\lambda_k u} du. \end{aligned}$$

Thus

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \sum_{k=1}^{\infty} e^{-\lambda_k u} du.$$

For  $\gamma > 0$ , the eigenvalues of  $\gamma A$  are  $\gamma \lambda_k$ , and doing the change of variables  $v = \gamma u$ ,

$$\begin{aligned} \zeta_{\gamma A}(s) &= \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \sum_{k=1}^{\infty} e^{-\gamma \lambda_k u} du \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} \gamma^{-s} v^{s-1} \sum_{k=1}^{\infty} e^{-\lambda_k v} dv \\ &= \gamma^{-s} \zeta_A(s). \end{aligned}$$

<sup>12</sup>cf. <http://individual.utoronto.ca/jordanbell/notes/fredholm.pdf>

<sup>13</sup>cf. <http://individual.utoronto.ca/jordanbell/notes/functionaldeterminant.pdf>



Taking the derivative,

$$\zeta'_{\gamma A}(s) = -\log \gamma \cdot \gamma^{-s} \cdot \zeta_A(s) + \gamma^{-s} \gamma'_A(s),$$

and then

$$\zeta'_{\gamma A}(0) = -\log \gamma \cdot \zeta_A(0) + \zeta'_A(0).$$

Then

$$\det(\gamma A) = e^{-\zeta'_{\gamma A}(0)} = e^{\log \gamma \cdot \zeta_A(0) - \zeta'_A(0)} = \gamma^{\zeta_A(0)} \det A.$$