# Gelfand-Pettis integrals and weak holomorphy

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## 1 Convexity

The Hahn-Banach separation theorem[1] states that if $A$ and $B$ are disjoint nonempty closed convex subsets of a locally convex space $X$ and $A$ is compact, then there is some $\lambda \in X^*$ and some $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\operatorname{Re} \lambda a < \gamma_1 < \gamma_2 < \operatorname{Re} \lambda b, \qquad a \in A, b \in B.$$

If $X$ is a vector space and $E$ is a subset of $X$, the *convex hull of* $E$ is defined to be the intersection of all convex sets containing $E$, and is denoted by $\operatorname{co}(E)$. One checks that the convex hull of $E$ is equal to the set of all finite convex combinations of elements of $E$. If $X$ is a topological vector space, the *closed convex hull of* $E$ is the intersection of all closed convex sets containing $E$, and is denoted by $\overline{\operatorname{co}}(E)$. The closed convex hull of $E$ is equal to the closure of the convex hull of $E$.[2]

## 2 The Gelfand-Pettis integral

If $X$ is a topological vector space over $F$, where $F$ is either $\mathbb{C}$ or $\mathbb{R}$, and $\mathscr{F}$ is a set of functions $X \to F$, we say that $\mathscr{F}$ *separates* $X$ if $x, y \in X$ being distinct implies that there is some $f \in \mathscr{F}$ satisfying $f(x) \neq f(y)$. It follows from the Hahn-Banach separation theorem that if $X$ is a locally convex space then its dual space $X^*$ separates $X$: if $a, b \in X$ are distinct then apply the Hahn-Banach separation to the sets $\{a\}$ and $\{b\}$.

Let $\mu$ be a positive measure on a measure space $Q$ and let $X$ be a topological vector space over $F$, where $F$ is either $\mathbb{C}$ or $\mathbb{R}$, such that $X^*$ separates $X$. If $f : Q \to X$ is a function and $\lambda \in X^*$, we define $\lambda f : Q \to F$ by $(\lambda f)(q) = \lambda f(q)$. If $\lambda f$ is integrable with respect to $\mu$ for each $\lambda \in X^*$ and there is some $I_f \in X$ such that

$$\lambda I_f = \int_Q \lambda f d\mu, \qquad \lambda \in X^*, \tag{1}$$

[1] Walter Rudin, *Functional Analysis*, second ed., p. 59, Theorem 3.4

[2] John B. Conway, *A Course in Functional Analysis*, second ed., p. 102, Corollary 1.13.

then we define

$$\int_Q f d\mu = I_f,$$

which we call the *Gelfand-Pettis integral of* $f$. Because $X^*$ separates $X$, there is at most one $I_f \in X$ satisfing (1), so if the Gelfand-Pettis integral of $f$ exists it is unique. If the Gelfand-Pettis integrals of $f$ and $g$ exist and $\alpha \in F$, then $\lambda(\alpha f + g)$ is integrable with respect to $\mu$ for each $\lambda \in X^*$, and

$$\begin{aligned}
\lambda(\alpha I_f + I_g) &= \alpha\lambda I_f + \lambda I_g \\
&= \alpha \int_Q \lambda f d\mu + \int_Q \lambda g d\mu \\
&= \int_Q \alpha\lambda f + \lambda g d\mu \\
&= \int_Q \lambda(\alpha f + g) d\mu,
\end{aligned}$$

therefore the Gelfand-Pettis integral of $\alpha f + g$ exists and satisfies

$$\int_Q \alpha f + g d\mu = \alpha \int_Q f d\mu + \int_Q g d\mu,$$

namely, Gelfand-Pettis integration is linear.

The following theorem gives conditions under which the Gelfand-Pettis integral of a function taking values in a real topological vector space exists.[3]

**Theorem 1.** *Suppose that*

- $X$ *is a real topological vector space such that* $X^*$ *separates* $X$
- $\mu$ *is a Borel probability measure on a compact Hausdorff space* $Q$

*If* $f : Q \to X$ *is continuous and* $\overline{\text{co}}(f(Q))$ *is a compact subset of* $X$*, then the Gelfand-Pettis integral*

$$\int_Q f d\mu$$

*exists, and* $\int_Q f d\mu \in \overline{\text{co}}(f(Q))$.

*Proof.* If $L$ is a finite subset of $X^*$, let $E_L$ be those $y \in \overline{\text{co}}(f(Q))$ such that

$$\lambda y = \int_Q \lambda f d\mu, \qquad \lambda \in L.$$

If $y_\alpha \in E_L$ is a net that converges to some $y \in \overline{\text{co}}(f(Q))$ and $\lambda \in L$, then, because $\lambda$ is continuous,

$$\lambda y = \lambda y_\alpha = \int_Q \lambda f d\mu,$$

[3] Walter Rudin, *Functional Analysis*, second ed., p. 78, Theorem 3.27. cf. Paul Garrett, *Vector-valued integrals*, `http://www.math.umn.edu/~garrett/m/fun/notes_2012-13/07e_vv_integrals.pdf`

showing that $y \in E_L$ and thus that $E_L$ is closed. By hypothesis $\overline{\text{co}}(f(Q))$ is compact, hence $E_L$ is compact. With $L = \{\lambda_1, \ldots, \lambda_n\}$, define $F_L \in \mathscr{B}(X, \mathbb{R}^n)$ by

$$F_L x = (\lambda_1 x, \ldots, \lambda_n x),$$

write $K = F_L f(Q)$, and define

$$m_i = \int_Q \lambda_i f d\mu, \qquad 1 \leq i \leq n.$$

Since $Q$ is compact and $f$ and $F_L$ are continuous, the set $K \subset \mathbb{R}^n$ is compact, and hence its convex hull $\text{co}(K)$ is compact.[4] If $t = (t_1, \ldots, t_n) \in \mathbb{R}^n \setminus \text{co}(K)$, then applying the Hahn-Banach separation theorem to $\text{co}(K)$ and $\{t\}$, we get that there is some $c \in (\mathbb{R}^n)^*$ and some $\gamma \in \mathbb{R}$ such that

$$ca < \gamma < ct, \qquad a \in \text{co}(K),$$

i.e. there is some $(c_1, \ldots, c_n) \in \mathbb{R}^n$ and some $\gamma \in \mathbb{R}$ such that

$$\sum_{i=1}^n c_i a_i < \gamma < \sum_{i=1}^n c_i t_i, \qquad a \in \text{co}(K).$$

If $q \in Q$ then $F_L f(q) \in K \subseteq \text{co}(K)$, hence

$$\sum_{i=1}^n c_i \lambda_i f(q) < \gamma < \sum_{i=1}^n c_i t_i, \qquad q \in Q.$$

Because $\mu$ is a probability measure, integrating the above inequality gives

$$\int_Q \sum_{i=1}^n c_i \lambda_i f(q) d\mu < \sum_{i=1}^n c_i t_i,$$

hence

$$\sum_{i=1}^n c_i m_i < \sum_{i=1}^n c_i t_i,$$

and it follows that $m \neq t$. Therefore $m \in \text{co}(K)$, and as $K = F_L f(Q)$, it follows that $m$ is a convex combination of finitely many points of the form $F_L f(q)$, i.e., $m$ is of the form $F_L y$ for some $y \in \text{co}(f(Q))$. To say that $m = F_L y$ means that

$$\lambda_i y = m_i = \int_Q \lambda_i f d\mu, \qquad 1 \leq i \leq n,$$

and thus $y \in E_L$. Therefore, if $L$ is a finite subset of $X^*$ then $E_L \neq \emptyset$.

If $\mathscr{S}$ is a set of sets, we say that $\mathscr{S}$ has the *finite intersection property* if $\mathscr{S}_0$ being a finite subset of $\mathscr{S}$ implies that $\bigcap_{S \in \mathscr{S}_0} S \neq \emptyset$. It is a fact that a topological space is compact if and only if every collection $\mathscr{S}$ of closed subsets

[4]Walter Rudin, *Functional Analysis*, second ed., p. 72, Theorem 3.20.

with the finite intersection property satisfies $\bigcap_{C\in\mathscr{C}} C \neq \emptyset$.[5] It follows from this that if $\mathscr{C}$ is a collection of compact subsets of a Hausdorff space and $\mathscr{C}$ has the finite intersection property, then $\bigcap_{C\in\mathscr{C}} C \neq \emptyset$. If $L, M$ are finite subsets of $X^*$, then $E_L \cap E_M = E_{L\cup M}$. We have shown that if $L$ is a finite subset of $X^*$ then $E_L \neq \emptyset$, and therefore the collection of all $E_L$, where $L$ is a finite subset of $X^*$, has the finite intersection property. As each $E_L$ is a compact set, we obtain

$$\bigcap_{L\subset X^*, |L|<\infty} E_L \neq \emptyset,$$

i.e. there is some $y \in \overline{\mathrm{co}}(f(Q))$ such that

$$\lambda y = \int_Q \lambda f d\mu, \qquad \lambda \in X^*.$$

This satisfies (1), so

$$y = \int_Q f d\mu,$$

which proves the claim. □

In a Fréchet space, the closed convex hull of a compact set is itself compact.[6] Thus, if $X$ is a Fréchet space then the set $\overline{\mathrm{co}}(f(Q))$ in the above theorem will be compact.

The following is the triangle inequality for Gelfand-Pettis integrals.[7]

**Corollary 2.** *If $Q$ is a compact Hausdorff space, $X$ is a real Banach space, $f : Q \to X$ is continuous, and $\mu$ is a Borel probability measure on $Q$, then*

$$\left\| \int_Q f d\mu \right\| \leq \int_Q \|f\| \, d\mu.$$

*Proof.* The Hahn-Banach extension theorem[8] states that if $X$ is a normed space and $x_0 \in X$, then there is some $\lambda \in X^*$ such that $\lambda x_0 = \|x_0\|$ and

$$|\lambda x| \leq \|x\|, \qquad x \in X.$$

Let $y = \int_Q f d\mu \in X$, and applying the Hahn-Banach extension theorem we get that there is some $\lambda \in X^*$ such that $\lambda y = \|y\|$ and $|\lambda x| \leq \|x\|$ for all $x \in X$. If $q \in Q$ then $f(q) \in X$, and so $|\lambda f(q)| \leq \|f(q)\|$ for all $q \in Q$, and integrating this inequality gives us

$$\int_Q |\lambda f(q)| d\mu \leq \int_Q \|f(q)\| \, d\mu.$$

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[5] James Munkres, *Topology*, second ed., p. 169, Theorem 26.9.
[6] Walter Rudin, *Functional Analysis*, second ed., p. 72, Theorem 3.20.
[7] Walter Rudin, *Functional Analysis*, second ed., p. 81, Theorem 3.29.
[8] Walter Rudin, *Functional Analysis*, second ed., p. 59, Corollary to Theorem 3.3.

Therefore,

$$\left\|\int_Q f d\mu\right\| = \|y\| = \lambda y = \int_Q \lambda f d\mu \leq \int_Q |\lambda f(q)| d\mu \leq \int_Q \|f(q)\| \, d\mu,$$

proving the claim. □

# 3 Holomorphy

A *path in* $\mathbb{C}$ is a continuous piecewise $C^1$ function from a compact interval to $\mathbb{C}$, and a *closed path* is a path whose initial point is equal to its final point. We denote by $\gamma^*$ the image of a path. If $\gamma : [\alpha, \beta] \to \mathbb{C}$ and $f : \gamma^* \to \mathbb{C}$ is a continuous function (where $\gamma^*$ has the subspace topology inherited from $\mathbb{C}$), the *contour integral of $f$ along* $\gamma$ is defined to be

$$\int_\gamma f(z) dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t) dt.$$

The *length of* $\gamma$, denoted $|\gamma|$, is defined to be $\int_\alpha^\beta |\gamma'(t)| dt$, and we have

$$\left|\int_\gamma f(z) dz\right| \leq \sup_{z \in \gamma^*} |f(z)| \cdot \int_\alpha^\beta |\gamma'(t)| dt.$$

If $\gamma$ is a closed path in $\mathbb{C}$ and $\Omega = \mathbb{C} \setminus \gamma^*$, we define

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}, \qquad z \in \Omega.$$

We call $\mathrm{Ind}_\gamma(z)$ the *index of $z$ with respect to* $\gamma$. It is a fact that $\mathrm{Ind}_\gamma$ takes integer values, is constant on each connected component of $\Omega$, and is 0 on the unique unbounded component of $\Omega$.[9]

Let $X$ is a complex topological vector space and let $\Omega$ be an open subset of $\mathbb{C}$. A function $f : \Omega \to X$ is said to be *weakly holomorphic in* $\Omega$ if $\lambda f : \Omega \to \mathbb{C}$ is holomorphic for every $\lambda \in X^*$, i.e., for every $\lambda \in X^*$ and $z \in \Omega$, the limit

$$\lim_{w \to z} \frac{(\lambda f)(w) - (\lambda f)(z)}{w - z}$$

exists. A function $f : \Omega \to X$ is said to be *strongly holomorphic* if for every $z \in \Omega$ the limit

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists. Check that a strongly holomorphic function is weakly holomorphic.

In the following theorem we show that if a function taking values in a complex locally convex topological vector space is weakly holomorphic then it is continuous.[10]

[9] Walter Rudin, *Real and Complex Analysis*, third ed., p. 203, Theorem 10.10; cf. Paul Garrett, *Holomorphic vector-valued functions*, `http://www.math.umn.edu/~garrett/m/fun/Notes/09_vv_holo.pdf`

[10] Walter Rudin, *Functional Analysis*, second ed., p. 82, Theorem 3.31(a).

**Theorem 3.** *If $\Omega$ is an open subset of $\mathbb{C}$, $X$ is a complex locally convex topological vector space space, and $f : \Omega \to X$ is weakly holomorphic, then $f : \Omega \to X$ is continuous.*

*Proof.* Let $\lambda \in X^*$. I assert that it suffices to prove the claim in the case that $0 \in \Omega$, and in this case just to prove that $f$ is continuous at 0.

Since $\Omega$ is an open set containing 0, there is some closed disc $\Delta_R$ of radius $R > 0$ with $0 \in \Delta_R \subset \Omega$. Define $\gamma_r = re^{it}$, $t \in [0, 2\pi]$, with $0 < r \leq R$. By assumption $\lambda f : \Omega \to \mathbb{C}$ is holomorphic, and applying Cauchy's integral formula,[11] if $z \in \Delta_r$ then

$$(\lambda f)(z)\mathrm{Ind}_{\gamma_r}(z) = \frac{1}{2\pi i}\int_{\gamma_r} \frac{(\lambda f)(\zeta)}{\zeta - z} d\zeta.$$

As $\mathrm{Ind}_{\gamma_r}(z) = 1$ for $z \in \Delta_r$, we have for every $z$ with $0 < |z| \leq r$ that

$$\begin{aligned}\frac{(\lambda f)(z) - (\lambda f)(0)}{z} &= \frac{1}{z} \cdot \frac{1}{2\pi i}\int_{\gamma_r} \frac{(\lambda f)(\zeta)}{\zeta - z} d\zeta - \frac{1}{z} \cdot \frac{1}{2\pi i}\int_{\gamma_r} \frac{(\lambda f)(\zeta)}{\zeta} d\zeta \\ &= \frac{1}{2\pi i}\int_{\gamma_r} \frac{(\lambda f)(\zeta)}{(\zeta - z)\zeta} d\zeta.\end{aligned}$$

Setting $M(\lambda) = \sup_{z \in \Delta_R} |(\lambda f)(z)|$, applying the above with $r = \frac{R}{2}$ we get that if $0 < |z| \leq \frac{R}{2}$ then

$$\left|\frac{(\lambda f)(z) - (\lambda f)(0)}{z}\right| \leq \frac{1}{2\pi} \cdot |\gamma_{\frac{R}{2}}| \cdot M(\lambda) \cdot \frac{1}{\inf_{|\zeta| = \frac{R}{2}} |\zeta - z||\zeta|} = \frac{2M(\lambda)}{r}.$$

Define

$$Y = \left\{ \frac{f(z) - f(0)}{z} : 0 < |z| \leq \frac{R}{2} \right\} \subseteq X.$$

We have shown that if $y \in Y$ and $\lambda \in X^*$, then

$$\lambda y = \lambda \frac{f(z) - f(0)}{z} = \frac{\lambda f(z) - \lambda f(0)}{z} = \frac{(\lambda f)(z) - (\lambda f)(0)}{z},$$

for some $0 < |z| \leq \frac{R}{2}$, hence

$$|\lambda y| \leq \frac{2M(\lambda)}{r}, \qquad y \in Y, \lambda \in X^*.$$

To say that a subset $E$ of $X$ is *weakly bounded* means that it is a bounded set in the weak topology on $X$, i.e. for every weak neighborhood $N$ of 0 there is some $c$ such that $E \subseteq cN$. $E$ is weakly bounded if and only if for every $\lambda \in X^*$ there is some constant $\gamma(\lambda)$ such that $x \in E$ implies that $|\lambda x| \leq \gamma(\lambda)$.[12] We have thus established that $Y$ is a weakly bounded subset of $X$. It is a fact that

[11] Walter Rudin, *Real and Complex Analysis*, third ed., p. 207, Theorem 10.15.

[12] cf. Walter Rudin, *Functional Analysis*, second ed., p. 66, §3.11.

a subset of a locally convex topological vector space is bounded if and only if it is weakly bounded,[13] so $Y$ is a bounded subset of $X$: if $N$ is a neighborhood of 0, then there is some $c$ such that $0 < |z| \leq r$ implies that

$$\frac{f(z) - f(0)}{z} \in cN.$$

Therefore if $0 < |z| \leq r \wedge \frac{1}{|c|}$ then

$$f(z) - f(0) \in N,$$

showing that $f$ is continuous at 0. □

**Theorem 4** (Cauchy integral formula). *If $\Omega$ is an open subset of $\mathbb{C}$, $X$ is a complex Fréchet space, $f : \Omega \to X$ is weakly holomorphic, $\gamma : [0,1] \to \Omega$ is a closed $C^1$ path, $z \notin \gamma^*$, and $\mathrm{Ind}_\gamma(z) = 1$, then the Gelfand-Pettis integral of $\frac{f \circ \gamma}{\gamma - z} \cdot \gamma' : [0,1] \to X$ exists and satisfies*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{[0,1]} \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t) dt.$$

*Proof.* By Theorem 3, $f : \Omega \to X$ is continuous. Because $\gamma^*$ is compact and $z \notin \gamma^*$, the function $t \mapsto \frac{1}{\gamma(t) - z}$ is continuous $[0,1] \to \mathbb{C}$. As $\gamma$ is $C^1$, the function $\gamma' : [0,1] \to \mathbb{C}$ is continuous. Thus $F(t) = \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t)$ continuous $[0,1] \to X$. We apply Theorem 1, which tells us that the Gelfand-Pettis integral of $F$ exists. Let

$$I = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{[0,1]} \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t) dt.$$

If $\lambda \in X^*$, then

$$\lambda I = \int_{[0,1]} \lambda \left( \frac{1}{2\pi i} F \right) dt = \frac{1}{2\pi i} \int_{[0,1]} \frac{\lambda f(\gamma(t))}{\gamma(t) - z} \gamma'(t) dt.$$

We apply the Cauchy integral formula for holomorphic functions on $\mathbb{C}$ to $\lambda f$:

$$(\lambda f)(z) = \frac{1}{2\pi i} \int_\gamma \frac{(\lambda f)(\zeta)}{\zeta - z} d\zeta.$$

Therefore $\lambda I = (\lambda f)(z) = \lambda(f(z))$. As this is true for all $\lambda \in X^*$, we have $I = f(z)$. □

One can use the above statement of the Cauchy integral formula to prove that a weakly holomorphic function that takes values in a complex Fréchet space is strongly holomorphic.[14]

**Theorem 5.** *If $\Omega$ is an open subset of $\mathbb{C}$, $X$ is a complex Fréchet space, and $f : \Omega \to X$ is weakly holomorphic, then $f$ is strongly holomorphic.*

[13] Walter Rudin, *Functional Analysis*, second ed., p. 70, Theorem 3.18.

[14] Walter Rudin, *Functional Analysis*, second ed., p. 82, Theorem 3.31(c); Paul Garrett, *Holomorphic vector-valued functions*, `http://www.math.umn.edu/~garrett/m/fun/notes_2012-13/08b_vv_holo.pdf`