The inhomogeneous heat equation on $\mathbb{T}$

Jordan Bell
jordan.bell@gmail.com
Department of Mathematics, University of Toronto

April 3, 2014

1 Introduction

In this note I am working out some material following Steve Shkoller's *MAT218: Lecture Notes on Partial Differential Equations*. However, I have written out a number of details that were not in the original notes, and may thus have introduced errors that were not in the notes on which this is based.

Write $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and for $1 \leq p < \infty$,

$$\|f\|_{L^p} = \left(\frac{1}{2\pi} \int_{\mathbb{T}} |f(t)|^p dt \right)^{1/p}.$$

Define

$$\|f\|_{H^k} = \left(\sum_{0 \leq j \leq k} \left\| \partial_x^j f \right\|_{L^2}^2 \right)^{1/2}.$$

If u is a distribution on $\mathbb{T}$, $\partial_x u$ is also a distribution on $\mathbb{T}$, and in particular, if $u \in L^2(\mathbb{T})$ then $\partial_x u$ is a distribution on $\mathbb{T}$. But if $u \in H^2(\mathbb{T})$, for example, then $\partial_x^2 u$ is an element of $L^2(\mathbb{T})$, rather than merely being a distribution.

Fix $T > 0$. Let $f \in L^2(0, T; L^2(\mathbb{T}))$ and $g \in H^1(\mathbb{T})$; as $H^1(\mathbb{T}) \subset C^0(\mathbb{T})$, we can speak about the value of g at every point rather than merely almost all points.

For almost all t and for all x, define f_n by

$$f_n(x, t) = \sum_{k=-n}^{n} \hat{f}(k, t) e^{ikx},$$

and for all x define g_n by

$$g_n(x) = \sum_{k=-n}^{n} \hat{g}(k) e^{ikx}.$$

In other words, if $D_n(x) = \sum_{k=-n}^{n} e^{ikx}$, then

$$f_n(x, t) = (D_n * f(\cdot, t))(x), \qquad g_n(x) = (D_n * g)(x),$$

where

$$(\phi * \psi)(x) = \frac{1}{2\pi} \int_{\mathbb{T}} \phi(y)\psi(x-y)dy.$$

2 Truncation

For each n, assume that there is some $u_n \in C^\infty(0,T;C^\infty(\mathbb{T}))$ such that for almost all t and for all $x \in \mathbb{T}$,

$$u_{nt}(x,t) - u_{nxx}(x,t) = f_n(x,t), \tag{1}$$

and for all $x \in \mathbb{T}$,

$$u_n(x,0) = g_n(x).$$

We will thus obtain a formula for u_n. In fact we will not necessarily have $u_n \in C^\infty(0,T;C^\infty(\mathbb{T}))$, but once we have an expression for u_n we can determine the function space of which it is an element. We will then show that there is some u in a certain function space such that $u_n(x,t) = (D_n * u(\cdot,t))(x)$ for all x and t.

For all t and x,

$$u_n(x,t) = \sum_{k\in\mathbb{Z}} \widehat{u_n}(k,t)e^{ikx}.$$

Then (1) becomes the statement that for almost all t and for all x,

$$\sum_{k\in\mathbb{Z}} \widehat{u_n}'(k,t)e^{ikx} + \sum_{k\in\mathbb{Z}} k^2\widehat{u_n}(k,t)e^{ikx} = \sum_{k=-n}^{n} \hat{f}(k,t)e^{ikx}.$$

If $|k| > n$, then $\widehat{u_n}'(k,t) + k^2\widehat{u_n}(k,t) = 0$, which is a linear ordinary differential equation, whose solution satisfies $\widehat{u_n}(k,t) = e^{-k^2t}\widehat{u_n}(k,0)$. Since $u_n(x,0) = g_n(x)$, $\widehat{u_n}(k,0) = 0$. Hence if $|k| > n$ then $\widehat{u_n}(k,t) = 0$. If $|k| \leq n$, then for almost all t, $\widehat{u_n}'(k,t) + k^2\widehat{u_n}(k,t) = \hat{f}(k,t)$. The solution of this is, for all t and for all x,

$$\widehat{u_n}(k,t) = e^{-k^2t}\widehat{g_n}(k) + e^{-k^2t}\int_0^t e^{k^2s}\widehat{f_n}(k,s)ds.$$

Hence, for all t and for all x,

$$u_n(x,t) = \sum_{k=-n}^{n} \left(e^{-k^2t}\widehat{g_n}(k) + e^{-k^2t}\int_0^t e^{k^2s}\widehat{f_n}(k,s)ds \right) e^{ikx}.$$

We merely know that $\widehat{f_n}(k,t)$ is defined for almost all t, thus we only know for almost all $t_0 \in (0,T)$ and for all x that $u_{nt}(x,t_0)$ exists. We do have that

$$u_n \in C^0(0,T;C^\infty(\mathbb{T})).$$

3 H^1

For almost all t, multiply (1) by $u_n(x,t)$ and integrate over $\mathbb{T}$. This is,

$$\int_{\mathbb{T}} u_{nt}(x,t)u_n(x,t)dx - \int_{\mathbb{T}} u_{nxx}(x,t)u_n(x,t)dx = \int_{\mathbb{T}} f_n(x,t)u_n(x,t)dx.$$

Integrating by parts this becomes

$$\int_{\mathbb{T}} u_{nt}(x,t)u_n(x,t)dx + \int_{\mathbb{T}} u_{nx}(x,t)u_{nx}(x,t)dx = \int_{\mathbb{T}} f_n(x,t)u_n(x,t)dx,$$

which is

$$\pi \cdot \partial_t \frac{1}{2\pi}\int_{\mathbb{T}} u_n(x,t)^2 dx + 2\pi \cdot \frac{1}{2\pi}\int_{\mathbb{T}} u_{nx}(x,t)^2 dx = \int_{\mathbb{T}} f_n(x,t)u_n(x,t)dx.$$

Writing this using norms,

$$\pi \cdot \partial_t \left\|u_n(\cdot,t)\right\|_{L^2}^2 + 2\pi \cdot \left\|u_{nx}(\cdot,t)\right\|_{L^2}^2 = \int_{\mathbb{T}} f_n(x,t)u_n(x,t)dx.$$

Integrating from 0 to t, for any t,

$$\pi\cdot\left\|u_n(\cdot,t)\right\|_{L^2}^2 - \pi\cdot\left\|u_n(\cdot,0)\right\|_{L^2}^2 + 2\pi\int_0^t \left\|u_{nx}(\cdot,s)\right\|_{L^2}^2 ds = \int_0^t\int_{\mathbb{T}} f_n(x,s)u_n(x,s)dxds.$$

For almost all s,

$$\begin{aligned}
\int_{\mathbb{T}} |f_n(x,s)u_n(x,s)|dx &= 2\pi \cdot \frac{1}{2\pi}\int_{\mathbb{T}} |f_n(x,s)u_n(x,s)|dx \\
&\le 2\pi \cdot \left\|f_n(\cdot,s)\right\|_{L^2}\left\|u_n(\cdot,s)\right\|_{L^2} \\
&\le 2\pi\left(\frac{\left\|f_n(\cdot,s)\right\|_{L^2}^2}{2} + \frac{\left\|u_n(\cdot,s)\right\|_{L^2}^2}{2}\right) \\
&= \pi \cdot \left\|f_n(\cdot,s)\right\|_{L^2}^2 + \pi \cdot \left\|u_n(\cdot,s)\right\|_{L^2}^2.
\end{aligned}$$

It follows that for all t (not just almost all t)

$$\begin{aligned}
&\pi \cdot \left\|u_n(\cdot,t)\right\|_{L^2}^2 - \pi \cdot \left\|u_n(\cdot,0)\right\|_{L^2}^2 + 2\pi\int_0^t \left\|u_{nx}(\cdot,s)\right\|_{L^2}^2 ds \\
&\le \int_0^t \pi \cdot \left\|f_n(\cdot,s)\right\|_{L^2}^2 + \pi \cdot \left\|u_n(\cdot,s)\right\|_{L^2}^2 ds,
\end{aligned}$$

so, as $u_n(x,0) = g_n(x)$,

$$\left\|u_n(\cdot,t)\right\|_{L^2}^2 + 2\int_0^t \left\|u_{nx}(\cdot,s)\right\|_{L^2}^2 ds \le \left\|g_n\right\|_{L^2}^2 + \int_0^t \left\|f_n(\cdot,s)\right\|_{L^2}^2 + \left\|u_n(\cdot,s)\right\|_{L^2}^2 ds.$$

Let

$$y(t) = \|u_n(\cdot,t)\|_{L^2}^2 + 2\int_0^t \|u_{nx}(\cdot,s)\|_{L^2}^2\,ds.$$

By the inequality we just established we have, for all t,

$$\begin{aligned} y(t) &\leq \|g_n\|_{L^2}^2 + \int_0^t \|f_n(\cdot,s)\|_{L^2}^2\,ds + \int_0^t \|u_n(\cdot,s)\|_{L^2}^2\,ds \\ &\leq \|g_n\|_{L^2}^2 + \int_0^t \|f_n(\cdot,s)\|_{L^2}^2\,ds + \int_0^t y(s)ds. \end{aligned}$$

By Gronwall's inequality, we get

$$y(t) \leq \left(\|g_n\|_{L^2}^2 + \int_0^t \|f_n(\cdot,s)\|_{L^2}^2\,ds\right)e^t.$$

As $\|g_n\|_{L^2} \leq \|g\|_{L^2}$ and $\|f_n(\cdot,s)\|_{L^2} \leq \|f(\cdot,s)\|_{L^2}$ (these two facts follow from Parseval's identity), it follows that

$$y(t) \leq \left(\|g\|_{L^2}^2 + \int_0^t \|f(\cdot,s)\|_{L^2}^2\,ds\right)e^t.$$

Therefore, if $0 \leq t \leq T$ then

$$\begin{aligned} \|u_n(\cdot,t)\|_{L^2}^2 + 2\int_0^t \|u_{nx}(\cdot,s)\|_{L^2}^2\,ds &\leq \left(\|g\|_{L^2}^2 + \int_0^t \|f(\cdot,s)\|_{L^2}^2\,ds\right)e^T \\ &\leq \left(\|g\|_{L^2}^2 + \|f\|_{L^2(0,T;L^2(\mathbb{T})}^2\right)e^T \\ &= M. \end{aligned}$$

By Parseval's identity,

$$\sum_{k\in\mathbb{Z}} |\widehat{u_n}(k,t)|^2 + 2\int_0^t \sum_{k\in\mathbb{Z}} |\widehat{u_{nx}}(k,s)|^2 ds \leq M,$$

hence for all t,

$$\sum_{k\in\mathbb{Z}} |\widehat{u_n}(k,t)|^2 + 2\int_0^t \sum_{k\in\mathbb{Z}} k^2|\widehat{u_n}(k,s)|^2 ds \leq M.$$

If $k \leq n \leq m$, then $\widehat{u_n}(k,t) = \widehat{u_m}(k,t)$ for all t. Define $\hat{u}(k,t)$ by

$$\hat{u}(k,t) = \lim_{n\to\infty} \widehat{u_n}(k,t) = \widehat{u_k}(k,t).$$

Thus, for all t,

$$\sum_{k\in\mathbb{Z}} |\hat{u}(k,t)|^2 + 2\int_0^t \sum_{k\in\mathbb{Z}} k^2|\hat{u}(k,s)|^2 ds \leq M. \tag{2}$$

Then, for some $M' = M'(f, g, T)$,

$$\int_0^T \sum_{k\in\mathbb{Z}} |\hat{u}(k,t)|^2 + \sum_{k\in\mathbb{Z}} k^2 |\hat{u}(k,t)|^2 dt \le M'.$$

It follows that for almost all t, there is some $u \in H^1(\mathbb{T})$ whose Fourier coefficients are $\hat{u}(k,t)$, and that we have

$$\int_0^T \|u(\cdot,t)\|_{H^1}^2 \, dt \le M'.$$

We have

$$\lim_{n\to\infty} \int_0^T \|u_n(\cdot,t) - u(\cdot,t)\|_{H^1}^2 \, dt = 0,$$

i.e.

$$\lim_{n\to\infty} \|u_n - u\|_{L^2(0,T;H^1(\mathbb{T}))}^2 = 0.$$

4 H^2

Multiply (1) by $u_{nxx}(x,t)$ and integrate over $\mathbb{T}$. We get, for almost all t,

$$\int_{\mathbb{T}} u_{nt}(x,t)u_{nxx}(x,t)dx - \int_{\mathbb{T}} u_{nxx}(x,t)u_{nxx}(x,t)dx = \int_{\mathbb{T}} f_n(x,t)u_{nxx}(x,t)dx.$$

As

$$\int_{\mathbb{T}} u_{nt}(x,t)u_{nxx}(x,t)dx = -\int_{\mathbb{T}} u_{ntx}(x,t)u_{nx}(x,t)dx = -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} u_{nx}(x,t)^2 dx,$$

we have

$$-\pi \frac{d}{dt} \|u_{nx}(\cdot,t)\|_{L^2}^2 - 2\pi \|u_{nxx}(\cdot,t)\|_{L^2}^2 = \int_{\mathbb{T}} f_n(x,t)u_{nxx}(x,t)dx.$$

Integrating from 0 to t,

$$\begin{aligned} & -\pi \|u_{nx}(\cdot,t)\|_{L^2}^2 + \pi \|u_{nx}(\cdot,0)\|_{L^2}^2 - 2\pi \int_0^t \|u_{nxx}(\cdot,s)\|_{L^2}^2 \, ds \\ = & \int_0^t \int_{\mathbb{T}} f_n(x,s)u_{nxx}(x,s)dx. \end{aligned}$$

For almost all s,

$$\begin{aligned} \int_{\mathbb{T}} |f_n(x,s)u_{nxx}(x,s)|dx &\le 2\pi \|f_n(\cdot,s)\|_{L^2} \|u_{nxx}(\cdot,s)\|_{L^2} \\ &\le 2\pi \left(\frac{\|f_n(\cdot,s)\|_{L^2}^2}{2} + \frac{\|u_{nxx}(\cdot,s)\|_{L^2}^2}{2} \right) \\ &= \pi \|f_n(\cdot,s)\|_{L^2}^2 + \pi \|u_{nxx}(\cdot,s)\|_{L^2}^2 . \end{aligned}$$

It follows that, for all t,

$$\begin{aligned}&\pi \left\|u_{nx}(\cdot,t)\right\|_{L^2}^2 x + 2\pi \int_0^t \left\|u_{nxx}(\cdot,s)\right\|_{L^2}^2 ds \\ \leq &\pi \left\|g_n'\right\|_{L^2}^2 + \pi \int_0^t \left\|f_n(\cdot,s)\right\|_{L^2}^2 + \left\|u_{nxx}(\cdot,s)\right\|_{L^2}^2 ds.\end{aligned}$$

Hence

$$\begin{aligned}\left\|u_{nx}(\cdot,t)\right\|_{L^2}^2 + \int_0^t \left\|u_{nxx}(\cdot,s)\right\|_{L^2}^2 ds &\leq \left\|g_n'\right\|_{L^2}^2 + \int_0^t \left\|f_n(\cdot,s)\right\|_{L^2}^2 ds \\ &\leq \left\|g\right\|_{H^1}^2 + \int_0^t \left\|f(\cdot,s)\right\|_{L^2}^2 ds \\ &\leq \left\|g\right\|_{H^1}^2 + \int_0^T \left\|f(\cdot,s)\right\|_{L^2}^2 ds.\end{aligned}$$

Using Parseval's identity we have, for all t,

$$\sum_{k\in\mathbb{Z}} |\widehat{u_{nx}}(k,t)|^2 + \int_0^t \sum_{k\in\mathbb{Z}} |\widehat{u_{nxx}}(k,s)|^2 ds \leq \left\|g\right\|_{H^1}^2 + \int_0^T \left\|f(\cdot,s)\right\|_{L^2}^2 ds,$$

hence

$$\sum_{k\in\mathbb{Z}} k^2 |\widehat{u_n}(k,t)|^2 + \int_0^t \sum_{k\in\mathbb{Z}} k^4 |\widehat{u_n}(k,s)|^2 ds \leq \left\|g\right\|_{H^1}^2 + \int_0^T \left\|f(\cdot,s)\right\|_{L^2}^2 ds,$$

so

$$\sum_{k\in\mathbb{Z}} k^2 |\widehat{u}(k,t)|^2 + \int_0^t \sum_{k\in\mathbb{Z}} k^4 |\widehat{u}(k,s)|^2 ds \leq \left\|g\right\|_{H^1}^2 + \int_0^T \left\|f(\cdot,s)\right\|_{L^2}^2 ds.$$

It follows that, for almost all t,[1]

$$\sum_{k\in\mathbb{Z}} k^2 |\widehat{u}(k,t)|^2 + \sum_{k\in\mathbb{Z}} k^4 |\widehat{u}(k,t)|^2 < \infty,$$

thus $u(\cdot,t) \in H^2(\mathbb{T})$.

We have

$$\lim_{n\to\infty} \int_0^T \left\|u_n(\cdot,t) - u(\cdot,t)\right\|_{H^2}^2 dt = 0,$$

i.e.

$$\lim_{n\to\infty} \left\|u_n - u\right\|_{L^2(0,T;H^2(\mathbb{T}))}^2 = 0.$$

[1] The reason I see that this follows involves the fact that the intersection of two sets of full measure is itself a set of full measure.

5 Solution

For all t we have $u(\cdot,t) \in H^1(\mathbb{T})$, and $H^1(\mathbb{T}) \subset C^0(\mathbb{T})$, so for all t and all x, $u(x,t)$ is defined. The Sobolev embedding tells us that if $k > \alpha + \frac{1}{2}$ then $H^k(\mathbb{T}) \subset C^\alpha(\mathbb{T})$. So, being specific, we have $H^1(\mathbb{T}) \subset C^{1/4}(\mathbb{T})$. It is a fact that if $h \in C^\alpha(\mathbb{T})$, $\alpha > 0$, then the partial sums of the Fourier series of h converge to h in the supremum norm.

For all t and for each k,

$$\hat{u}(k,t) = e^{-k^2 t}\widehat{g}(k) + e^{-k^2 t}\int_0^t e^{k^2 s}\widehat{f}(k,s)ds.$$

It follows that, for all x,

$$u(x,0) = \lim_{N\to\infty} \sum_{|k|\le N} \widehat{g}(k)e^{ikx}.$$

On the other hand,

$$g(x) = \lim_{N\to\infty} \sum_{|k|\le N} \widehat{g}(k)e^{ikx}.$$

Thus for all x, $u(x,0) = g(x)$.

We have

$$\begin{aligned}\|u_t - u_{xx} - f\|_{L^2(0,T;L^2(\mathbb{T}))} \le\; & \|u_t - u_{nt}\|_{L^2(0,T;L^2(\mathbb{T}))} \\ & + \|u_{xx} - u_{nxx}\|_{L^2(0,T;L^2(\mathbb{T}))} \\ & + \|f - f_n\|_{L^2(0,T;L^2(\mathbb{T}))} \\ & + \|u_{nt} - u_{nxx} - f_n\|_{L^2(0,T;L^2(\mathbb{T}))}.\end{aligned}$$

Each of the four norms has limit 0 as $n \to \infty$. Let me work out the first one. For almost all t,

$$\begin{aligned}\widehat{u_t}(k,t) - \widehat{u_{nt}}(k,t) &= \sum_{|k|>n} -k^2 e^{-k^2 t}\hat{g}(k) - k^2 e^{-k^2 t}\int_0^t e^{k^2 s}\hat{f}(k,s)ds \\ &\quad + e^{-k^2 t}e^{k^2 t}\hat{f}(k,t) \\ &= \sum_{|k|>n} -k^2\hat{u}(k,t) + \hat{f}(k,t).\end{aligned}$$

Then using Parseval's identity,

$$\begin{aligned}\|u_t - u_{nt}\|^2_{L^2(0,T;L^2(\mathbb{T}))} &= \int_0^T \sum_{|k|>n} |-k^2\hat{u}(k,t) + \hat{f}(k,t)|^2 dt \\ &\le 2\int_0^T \sum_{|k|>n} k^2|\hat{u}(k,t)|^2 + |\hat{f}(k,t)|^2 dt.\end{aligned}$$

Then,

$$\|u_t - u_{xx} - f\|^2_{L^2(0,T;L^2(\mathbb{T}))} = 0.$$

So, for almost all t and for almost all x,

$$u_t(x,t) - u_{xx}(x,t) = f(x,t).$$