

# The heat kernel on $\mathbb{R}^n$

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## 1 Notation

For  $f \in L^1(\mathbb{R}^n)$ , we define  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{C}$  by

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi x} dx, \quad \xi \in \mathbb{R}^n.$$

The statement of the Riemann-Lebesgue lemma is that  $\hat{f} \in C_0(\mathbb{R}^n)$ .

We denote by  $\mathcal{S}_n$  the Fréchet space of Schwartz functions  $\mathbb{R}^n \rightarrow \mathbb{C}$ .

If  $\alpha$  is a multi-index, we define

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n},$$

$$D_\alpha = i^{-|\alpha|} D^\alpha = \left(\frac{1}{i} D_1\right)^{\alpha_1} \cdots \left(\frac{1}{i} D_n\right)^{\alpha_n},$$

and

$$\Delta = D_1^2 + \cdots + D_n^2.$$

## 2 The heat equation

Fix  $n$ , and for  $t > 0$ ,  $x \in \mathbb{R}^n$ , define

$$k_t(x) = k(t, x) = (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right).$$

We call  $k$  the **heat kernel**. It is straightforward to check for any  $t > 0$  that  $k_t \in \mathcal{S}_n$ . The heat kernel satisfies

$$k_t(x) = (t^{-1/2})^n k_1(t^{-1/2}x), \quad t > 0, x \in \mathbb{R}^n.$$

For  $a > 0$  and  $f(x) = e^{-\pi a|x|^2}$ , it is a fact that  $\hat{f}(\xi) = a^{-n/2} e^{-\pi|\xi|^2/a}$ . Using this, for any  $t > 0$  we get

$$\hat{k}_t(\xi) = e^{-4\pi^2|\xi|^2 t}, \quad \xi \in \mathbb{R}^n.$$

Thus for any  $t > 0$ ,

$$\int_{\mathbb{R}^n} k_t(x) dx = \hat{k}_t(0) = 1.$$

Then the heat kernel is an **approximate identity**: if  $f \in L^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ , then  $\|f * k_t - f\|_p \rightarrow 0$  as  $t \rightarrow 0$ , and if  $f$  is a function on  $\mathbb{R}^n$  that is bounded and continuous, then for every  $x \in \mathbb{R}^n$ ,  $f * k_t(x) \rightarrow f(x)$  as  $t \rightarrow 0$ .<sup>1</sup> For each  $t > 0$ , because  $k_t \in \mathcal{S}_n$  we have  $f * k_t \in C^\infty(\mathbb{R}^n)$ , and  $D^\alpha(f * k_t) = f * D^\alpha k_t$  for any multi-index  $\alpha$ .<sup>2</sup>

The **heat operator** is  $D_t - \Delta$  and the **heat equation** is  $(D_t - \Delta)u = 0$ . It is straightforward to check that

$$(D_t - \Delta)k(t, x) = 0, \quad t > 0, x \in \mathbb{R}^n,$$

that is, the heat kernel is a solution of the heat equation.

To get some practice proving things about solutions of the heat equation, we work out the following theorem from Folland.<sup>3</sup> In Folland's proof it is not apparent how the hypotheses on  $u$  and  $D_x$  are used, and we make this explicit.

**Theorem 1.** *Suppose that  $u : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{C}$  is continuous, that  $u$  is  $C^2$  on  $(0, \infty) \times \mathbb{R}^n$ , that*

$$(D_t - \Delta)u(t, x) = 0, \quad t > 0, x \in \mathbb{R}^n,$$

and that  $u(0, x) = 0$  for  $x \in \mathbb{R}^n$ . If for every  $\epsilon > 0$  there is some  $C$  such that

$$|u(t, x)| \leq Ce^{\epsilon|x|^2}, \quad |D_x u(t, x)| \leq Ce^{\epsilon|x|^2}, \quad t > 0, x \in \mathbb{R}^n,$$

then  $u = 0$ .

*Proof.* If  $f$  and  $g$  are  $C^2$  functions on some open set in  $\mathbb{R} \times \mathbb{R}^n$ , such as  $(0, \infty) \times \mathbb{R}^n$ , then

$$\begin{aligned} g(\partial_t f - \Delta f) + f(\partial_t g + \Delta g) &= \partial_t(fg) - g \sum_{j=1}^n \partial_j^2 f + f \sum_{j=1}^n \partial_j^2 g \\ &= \partial_t(fg) + \sum_{j=1}^n \partial_j(f \partial_j g - g \partial_j f) \\ &= \operatorname{div}_{t,x} F, \end{aligned}$$

where

$$F = (fg, f \partial_1 g - g \partial_1 f, \dots, f \partial_n g - g \partial_n f).$$

<sup>1</sup> $k_1$ , and any  $k_t$ , belong merely to  $\mathcal{S}_n$  and not to  $\mathcal{D}(\mathbb{R}^n)$ , which is demanded in the definition of an approximate identity in Rudin's *Functional Analysis*, second ed.

<sup>2</sup>Gerald B. Folland, *Introduction to Partial Differential Equations*, second ed., p. 11, Theorem 0.14.

<sup>3</sup>Gerald B. Folland, *Introduction to Partial Differential Equations*, second ed., p. 144, Theorem 4.4.

Take  $t_0 > 0$ ,  $x_0 \in \mathbb{R}^n$ , and let  $f(t, x) = u(t, x)$  and  $g(t, x) = k(t_0 - t, x - x_0)$  for  $t > 0$ ,  $x \in \mathbb{R}^n$ . Let  $0 < a < b < t_0$  and  $r > 0$ , and define

$$\Omega = \{(t, x) : |x| < r, a < t < b\}.$$

In  $\Omega$  we check that  $(\partial_t - \Delta)f = 0$  and  $(\partial_t + \Delta)g = 0$ , so by the divergence theorem,

$$\int_{\partial\Omega} F \cdot \nu = \int_{\Omega} \operatorname{div}_{t,x} F = \int_{\Omega} g(\partial_t f - \Delta f) + f(\partial_t g + \Delta g) = \int_{\Omega} g \cdot 0 + f \cdot 0 = 0.$$

On the other hand, as

$$\partial\Omega = \{(b, x) : |x| \leq r\} \cup \{(a, x) : |x| \leq r\} \cup \{(t, x) : a < t < b, |x| = r\},$$

we have

$$\begin{aligned} \int_{\partial\Omega} F \cdot \nu &= \int_{|x| \leq r} F(b, x) \cdot (1, 0, \dots, 0) dx + \int_{|x| \leq r} F(a, x) \cdot (-1, 0, \dots, 0) dx \\ &\quad + \int_a^b \int_{|x|=r} F(t, x) \cdot \frac{x}{r} d\sigma(x) t^{n-1} dt \\ &= \int_{|x| \leq r} f(b, x)g(b, x) dx - \int_{|x| \leq r} f(a, x)g(a, x) dx \\ &\quad + \int_a^b \int_{|x|=r} \sum_{j=1}^n (f \partial_j g - g \partial_j f)(t, x) \frac{x_j}{r} d\sigma(x) t^{n-1} dt \\ &= \int_{|x| \leq r} u(b, x)k(t_0 - b, x - x_0) dx - \int_{|x| \leq r} u(a, x)k(t_0 - a, x - x_0) dx \\ &\quad + \int_a^b \int_{|x|=r} \sum_{j=1}^n \left( u(t, x) \partial_j k(t_0 - t, x - x_0) \right. \\ &\quad \left. - k(t_0 - t, x - x_0) \partial_j u(t, x) \right) \frac{x_j}{r} d\sigma(x) t^{n-1} dt, \end{aligned}$$

where  $\sigma$  is surface measure on  $\{|x| = r\} = rS^{n-1}$ . As  $r \rightarrow \infty$ , the first two terms tend to

$$\int_{\mathbb{R}^n} u(b, x)k_{t_0-b}(x - x_0) dx = \int_{\mathbb{R}^n} u(b, x)k_{t_0-b}(x_0 - x) dx = u(b, \cdot) * k_{t_0-b}(x_0)$$

and

$$\int_{\mathbb{R}^n} u(a, x)k_{t_0-a}(x - x_0) dx = \int_{\mathbb{R}^n} u(a, x)k_{t_0-a}(x_0 - x) dx = u(a, \cdot) * k_{t_0-a}(x_0)$$

respectively. Let  $\epsilon < \frac{1}{4(t_0-a)}$ , and let  $C$  be as given in the statement of the theorem. Using  $\partial_j k(t, x) = -\frac{x_j}{2t} k(t, x)$ , for any  $r > 0$  the third term is bounded by

$$n \int_a^b \int_{|x|=r} \left( C e^{\epsilon r^2} \frac{|x - x_0|}{2t} k(t_0 - t, x - x_0) + k(t_0 - t, x - x_0) C e^{\epsilon r^2} \right) d\sigma(x) t^{n-1} dt,$$

which is bounded by

$$n \int_a^b \int_{|x|=r} C e^{\epsilon r^2} \left( \frac{|x_0| + r}{2a} + 1 \right) (4\pi(t_0 - b))^{-n/2} \exp\left(-\frac{r^2}{4(t_0 - a)}\right) d\sigma(x) t^{n-1} dt,$$

and writing  $\eta = \frac{1}{4(t_0 - a)} - \epsilon$  and  $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ , the surface area of the sphere of radius 1 in  $\mathbb{R}^n$ , this is equal to

$$(b - a)^n r^{n-1} \omega_n C e^{-\eta r^2} \left( \frac{|x_0| + r}{2a} + 1 \right) (4\pi(t_0 - b))^{-n/2},$$

which tends to 0 as  $r \rightarrow \infty$ . Therefore,

$$u(b, \cdot) * k_{t_0 - b}(x_0) = u(a, \cdot) * k_{t_0 - a}(x_0).$$

One checks that as  $b \rightarrow t_0$ , the left-hand side tends to  $u(t_0, x_0)$ , and that as  $a \rightarrow 0$ , the right-hand side tends to  $u(0, x_0) = 0$ . Therefore,

$$u(t_0, x_0) = 0.$$

This is true for any  $t_0 > 0$ ,  $x_0 \in \mathbb{R}^n$ , and as  $u : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{C}$  is continuous, it follows that  $u$  is identically 0.  $\square$

### 3 Fundamental solutions

We extend  $k$  to  $\mathbb{R} \times \mathbb{R}^n$  as

$$k(t, x) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right) & t > 0, x \in \mathbb{R}^n \\ 0 & t \leq 0, x \in \mathbb{R}^n. \end{cases}$$

This function is locally integrable in  $\mathbb{R} \times \mathbb{R}^n$ , so it makes sense to define  $\Lambda_k \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^n)$  by

$$\Lambda_k \phi = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \phi(t, x) k(t, x) dx dt, \quad \phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^n).$$

Suppose that  $P$  is a polynomial in  $n$  variables:

$$P(\xi) = \sum c_\alpha \xi^\alpha = \sum c_\alpha \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}.$$

We say that  $E \in \mathcal{D}'(\mathbb{R}^n)$  is a **fundamental solution** of the differential operator

$$P(D) = \sum c_\alpha D_\alpha = \sum c_\alpha i^{-|\alpha|} D^\alpha$$

if  $P(D)E = \delta$ . If  $E = \Lambda_f$  for some locally integrable  $f$ ,  $\Lambda_f \phi = \int_{\mathbb{R}^n} \phi(x) f(x) dx$ , we also say that the function  $f$  is a fundamental solution of the differential operator  $P(D)$ . We now prove that the heat kernel extended to  $\mathbb{R} \times \mathbb{R}^n$  in the above way is a fundamental solution of the heat operator.<sup>4</sup>

<sup>4</sup>Gerald B. Folland, *Introduction to Partial Differential Equations*, second ed., p. 146, Theorem 4.6.

**Theorem 2.**  $\Lambda_k$  is a fundamental solution of  $D_t - \Delta$ .

*Proof.* For  $\epsilon > 0$ , define  $K_\epsilon(t, x) = k(t, x)$  if  $t > \epsilon$  and  $K_\epsilon(t, x) = 0$  otherwise. For any  $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^n)$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}} \int_{\mathbb{R}^n} (k(t, x) - K_\epsilon(t, x)) \phi(t, x) dx dt \right| &= \left| \int_0^\epsilon \int_{\mathbb{R}^n} k(t, x) \phi(t, x) dx dt \right| \\ &\leq \|\phi\|_\infty \int_0^\epsilon \int_{\mathbb{R}^n} k(t, x) dx dt \\ &= \|\phi\|_\infty \int_0^\epsilon dt \\ &= \|\phi\|_\infty \epsilon. \end{aligned}$$

This shows that  $\Lambda_{K_\epsilon} \rightarrow \Lambda_k$  in  $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^n)$ , with the weak-\* topology. It is a fact that for any multi-index,  $E \mapsto D^\alpha E$  is continuous  $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R} \times \mathbb{R}^n)$ , and hence  $(D_t - \Delta)\Lambda_{K_\epsilon} \rightarrow (D_t - \Delta)\Lambda_k$  in  $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^n)$ . Therefore, to prove the theorem it suffices to prove that  $(D_t - \Delta)\Lambda_{K_\epsilon} \rightarrow \delta$  (because  $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^n)$  with the weak-\* topology is Hausdorff).

Let  $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^n)$ . Doing integration by parts,

$$\begin{aligned} (D_t - \Delta)\Lambda_{K_\epsilon}(\phi) &= \Lambda_{K_\epsilon}((D_t - \Delta)\phi) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} K_\epsilon(t, x) (D_t \phi(t, x) - \Delta \phi(t, x)) dx dt \\ &= \int_\epsilon^\infty \int_{\mathbb{R}^n} k(t, x) D_t \phi(t, x) - k(t, x) \Delta \phi(t, x) dx dt \\ &= \int_{\mathbb{R}^n} \left( k(\epsilon, x) \phi(\epsilon, x) - \int_\epsilon^\infty \phi(t, x) D_t k(t, x) dt \right) dx \\ &\quad + \int_\epsilon^\infty \int_{\mathbb{R}^n} \phi(t, x) \Delta k(t, x) dx dt \\ &= \int_{\mathbb{R}^n} k(\epsilon, x) \phi(\epsilon, x) dx \\ &\quad - \int_\epsilon^\infty \int_{\mathbb{R}^n} \phi(t, x) (D_t - \Delta) k(t, x) dt dx \\ &= \int_{\mathbb{R}^n} k(\epsilon, x) \phi(\epsilon, x) dx. \end{aligned}$$

So, using  $k_t(x) = k_t(-x)$  and writing  $\phi_t(x) = \phi(t, x)$ ,

$$\begin{aligned} (D_t - \Delta)\Lambda_{K_\epsilon}(\phi) &= \int_{\mathbb{R}^n} k_\epsilon(-x) \phi_\epsilon(x) dx \\ &= k_\epsilon * \phi_\epsilon(0) \\ &= k_\epsilon * \phi_0(0) + k_\epsilon * (\phi_\epsilon - \phi_0)(0). \end{aligned}$$

Using the definition of convolution, the second term is bounded by

$$\sup_{x \in \mathbb{R}^n} |\phi_\epsilon(x) - \phi_0(x)| \|k_\epsilon\|_1 = \sup_{x \in \mathbb{R}^n} |\phi_\epsilon(x) - \phi_0(x)|,$$

which tends to 0 as  $\epsilon \rightarrow 0$ . Because  $k$  is an approximate identity,  $k_\epsilon * \phi_0(0) \rightarrow \phi_0(0)$  as  $\epsilon \rightarrow 0$ . That is,

$$(D_t - \Delta)\Lambda_{K_\epsilon}(\phi) \rightarrow \phi_0(0) = \delta(\phi)$$

as  $\epsilon \rightarrow 0$ , showing that  $(D_t - \Delta)\Lambda_{K_\epsilon} \rightarrow \delta$  in  $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^n)$  and completing the proof.  $\square$

## 4 Functions of the Laplacian

This section is my working through of material in Folland.<sup>5</sup> For  $f \in \mathcal{S}_n$  and for any nonnegative integer  $k$ , doing integration by parts we get

$$\mathcal{F}((-\Delta)^k f)(\xi) = \int_{\mathbb{R}^n} ((-\Delta)^k f)(x) e^{-2\pi i \xi x} dx = (4\pi^2 |\xi|^2)^k (\mathcal{F} f)(\xi), \quad \xi \in \mathbb{R}^n.$$

Suppose that  $P$  is a polynomial in one variable:  $P(x) = \sum c_k x^k$ . Then, writing  $P(-\Delta) = \sum c_k (-\Delta)^k$ , we have

$$\begin{aligned} \mathcal{F}(P(-\Delta)f)(\xi) &= \sum c_k \mathcal{F}((-\Delta)^k f)(\xi) \\ &= \sum c_k (4\pi^2 |\xi|^2)^k (\mathcal{F} f)(\xi) \\ &= (\mathcal{F} f)(\xi) P(4\pi^2 |\xi|^2). \end{aligned}$$

We remind ourselves that **tempered distributions** are elements of  $\mathcal{S}'_n$ , i.e. continuous linear maps  $\mathcal{S}_n \rightarrow \mathbb{C}$ . The Fourier transform of a tempered distribution  $\Lambda$  is defined by  $\hat{\Lambda} f = (\mathcal{F}\Lambda)f = \Lambda \hat{f}$ ,  $f \in \mathcal{S}_n$ . It is a fact that the Fourier transform is an isomorphism of locally convex spaces  $\mathcal{S}'_n \rightarrow \mathcal{S}'_n$ .<sup>6</sup>

Suppose that  $\psi : (0, \infty) \rightarrow \mathbb{C}$  is a function such that

$$\Lambda f = \int_{\mathbb{R}^n} f(\xi) \psi(4\pi^2 |\xi|^2) d\xi, \quad f \in \mathcal{S}_n,$$

is a tempered distribution. We define  $\psi(-\Delta) : \mathcal{S}_n \rightarrow \mathcal{S}'_n$  by

$$\psi(-\Delta)f = \mathcal{F}^{-1}(\hat{f}\Lambda), \quad f \in \mathcal{S}_n.$$

Define  $\check{f}(x) = f(-x)$ ; this is **not** the inverse Fourier transform of  $f$ , which we denote by  $\mathcal{F}^{-1}$ . As well, write  $\tau_x f(y) = f(y - x)$ . For  $u \in \mathcal{S}'_n$  and  $\phi \in \mathcal{S}_n$ , we **define** the convolution  $u * \phi : \mathbb{R}^n \rightarrow \mathbb{C}$  by

$$(u * \phi)(x) = u(\tau_x \check{\phi}), \quad x \in \mathbb{R}^n.$$

One proves that  $u * \phi \in C^\infty(\mathbb{R}^n)$ , that

$$D^\alpha(u * \phi) = (D^\alpha u) * \phi = u * (D^\alpha \phi)$$

<sup>5</sup>Gerald B. Folland, *Introduction to Partial Differential Equations*, second ed., pp. 149–152, §4B.

<sup>6</sup>Walter Rudin, *Functional Analysis*, second ed., p. 192, Theorem 7.15.

for any multi-index, that  $u * \phi$  is a tempered distribution, that  $\mathcal{F}(u * \phi) = \hat{\phi}\hat{u}$ , and that  $\hat{u} * \hat{\phi} = \mathcal{F}(\phi u)$ .<sup>7</sup>

We can also write  $\psi(-\Delta)$  in the following way. There is a unique  $\kappa_\psi \in \mathcal{S}'_n$  such that

$$\mathcal{F}\kappa_\psi = \Lambda.$$

For  $f \in \mathcal{S}_n$ , we have  $\mathcal{F}(\kappa_\psi * f) = \hat{f}\hat{\kappa}_\psi = \hat{f}\Lambda$ , but, using the definition of  $\psi(-\Delta)$  we also have  $\mathcal{F}(\psi(-\Delta)f) = \mathcal{F}\mathcal{F}^{-1}(\hat{f}\Lambda) = \hat{f}\Lambda$ , so

$$\kappa_\psi * f = \psi(-\Delta)f.$$

Moreover,  $\kappa_\psi * f \in C^\infty(\mathbb{R}^n)$ ; this shows that  $\psi(-\Delta)f$  can be interpreted as a tempered distribution or as a function. We call  $\kappa_\psi$  the **convolution kernel** of  $\psi(-\Delta)$ .

For a fixed  $t > 0$ , define  $\psi(s) = e^{-ts}$ . Then  $\Lambda : \mathcal{S}_n \rightarrow \mathbb{C}$  defined by

$$\Lambda f = \int_{\mathbb{R}^n} f(\xi)\psi(4\pi^2|\xi|^2)d\xi = \int_{\mathbb{R}^n} f(\xi)\exp(-4\pi^2|\xi|^2t)d\xi = \int_{\mathbb{R}^n} f(\xi)\hat{k}_t(\xi)d\xi$$

is a tempered distribution. Using the Plancherel theorem, we have

$$\Lambda f = \int_{\mathbb{R}^n} \hat{f}(\xi)k_t(\xi)d\xi.$$

With  $\kappa_\psi \in \mathcal{S}'_n$  such that  $\mathcal{F}\kappa_\psi = \Lambda$ , we have

$$\Lambda f = (\mathcal{F}\kappa_\psi)(f) = \kappa_\psi(\hat{f}).$$

Because  $f \mapsto \hat{f}$  is a bijection  $\mathcal{S}_n \rightarrow \mathcal{S}_n$ , this shows that for any  $f \in \mathcal{S}_n$  we have

$$\kappa_\psi(f) = \int_{\mathbb{R}^n} f(\xi)k_t(\xi)d\xi.$$

Hence,

$$e^{t\Delta}f = \kappa_\psi * f = k_t * f, \quad t > 0, f \in \mathcal{S}_n. \quad (1)$$

Suppose that  $\phi : (0, \infty) \rightarrow \mathbb{C}$  and  $\omega : (0, \infty) \rightarrow (0, \infty)$  are functions and that

$$\psi(s) = \int_0^\infty \phi(\tau)e^{-s\omega(\tau)}d\tau, \quad s > 0.$$

Manipulating symbols suggests that it may be true that

$$\psi(-\Delta) = \int_0^\infty \phi(\tau)e^{\omega(\tau)\Delta}d\tau,$$

and then, for  $f \in \mathcal{S}_n$ ,

$$\psi(-\Delta)f = \int_0^\infty \phi(\tau)e^{\omega(\tau)\Delta}fd\tau = \int_0^\infty \phi(\tau)(k_{\omega(\tau)} * f)d\tau,$$

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<sup>7</sup>Walter Rudin, *Functional Analysis*, second ed., p. 195, Theorem 7.19.

and hence

$$\kappa_\psi(x) = \int_0^\infty \phi(\tau)k_{\omega(\tau)}(x)d\tau, \quad x \in \mathbb{R}^n. \quad (2)$$

Take  $\psi(s) = s^{-\beta}$  with  $0 < \operatorname{Re} \beta < \frac{n}{2}$ . Because  $\operatorname{Re} \beta < \frac{n}{2}$ , one checks that

$$\Lambda f = \int_{\mathbb{R}^n} f(\xi)(4\pi^2|\xi|^2)^{-\beta}d\xi$$

is a tempered distribution. As  $\operatorname{Re} \beta > 0$ , we have

$$s^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-s\tau} d\tau$$

and writing  $\phi(\tau) = \frac{\tau^{\beta-1}}{\Gamma(\beta)}$  and  $\omega(\tau) = \tau$ , we suspect from (2) that the convolution kernel of  $(-\Delta)^{-\beta}$  is

$$\kappa_\psi(x) = \int_0^\infty \frac{\tau^{\beta-1}}{\Gamma(\beta)} k_\tau(x) d\tau,$$

which one calculates is equal to

$$\frac{\Gamma\left(\frac{n}{2} - \beta\right)}{\Gamma(\beta)4^\beta\pi^{n/2}|x|^{n-2\beta}}. \quad (3)$$

What we have written so far does not prove that this is the convolution kernel of  $(-\Delta)^{-\beta}$  because it used (2), but it is straightforward to calculate that indeed the convolution kernel of  $(-\Delta)^{-\beta}$  is (3). This calculation is explained in an exercise in Folland.<sup>8</sup>

Taking  $\alpha = 2\beta$  and defining

$$R_\alpha(x) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)2^\alpha\pi^{n/2}|x|^{n-\alpha}}, \quad 0 < \operatorname{Re} \alpha < n, x \in \mathbb{R}^n,$$

we call  $R_\alpha$  the **Riesz potential** of order  $\alpha$ . Taking as granted that (3) is the convolution kernel of  $(-\Delta)^{-\beta}$ , we have

$$(-\Delta)^{-\alpha/2}f = R_\alpha * f, \quad f \in \mathcal{S}_n.$$

Then, if  $n > 2$  and  $\alpha = 2$  satisfies  $0 < \operatorname{Re} \alpha < n$ , we work out that

$$R_2(x) = \frac{1}{(n-2)\omega_n|x|^{n-2}},$$

where  $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ , and hence

$$(-\Delta)^{-1}f = R_2 * f, \quad f \in \mathcal{S}_n,$$

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<sup>8</sup>Gerald B. Folland, *Introduction to Partial Differential Equations*, second ed., p. 154, Exercise 1.



and applying  $-\Delta$  we obtain

$$f = -\Delta(R_2 * f) = (-\Delta R_2) * f,$$

hence  $-\Delta R_2 = \delta$ . That is,  $R_2$  is the fundamental solution for  $-\Delta$ .

Suppose that  $\operatorname{Re} \beta > 0$ . Then, using the definition of  $\Gamma(\beta)$  as an integral, with  $\psi(s) = (1+s)^{-\beta}$ , we have

$$\psi(s) = \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-(1+s)\tau} d\tau, \quad s > 0.$$

Manipulating symbols suggests that

$$\psi(-\Delta) = \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-\tau} e^{\tau\Delta} d\tau,$$

and using (1), assuming the above is true we would have for all  $f \in \mathcal{S}_n$ ,

$$\psi(-\Delta)f = \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-\tau} e^{\tau\Delta} f d\tau = \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-\tau} (k_\tau * f) d\tau,$$

whose convolution kernel is

$$\frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-\tau} k_\tau d\tau.$$

We write  $\alpha = 2\beta$  and define, for  $\operatorname{Re} \alpha > 0$ ,

$$B_\alpha(x) = \frac{1}{\Gamma(\frac{\alpha}{2}) (4\pi)^{n/2}} \int_0^\infty \tau^{\frac{\alpha-n}{2}-1} e^{-\tau - \frac{|x|^2}{4\tau}} d\tau, \quad x \neq 0.$$

We call  $B_\alpha$  the **Bessel potential** of order  $\alpha$ . It is straightforward to show, and shown in Folland, that  $\|B_\alpha\|_1 < \infty$ , so  $B_\alpha \in L^1(\mathbb{R}^n)$ . Therefore we can take the Fourier transform of  $B_\alpha$ , and one calculates that it is

$$\widehat{B}_\alpha(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}, \quad \xi \in \mathbb{R}^n,$$

and then

$$\psi(-\Delta) = (1 - \Delta)^{-\alpha/2} f = B_\alpha * f, \quad f \in \mathcal{S}_n.$$

## 5 Gaussian measure

If  $\mu$  is a measure on  $\mathbb{R}^n$  and  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  is a function such that for every  $x \in \mathbb{R}^n$  the integral  $\int_{\mathbb{R}^n} f(x-y) d\mu(y)$  converges, we define the **convolution**  $\mu * f : \mathbb{R}^n \rightarrow \mathbb{C}$  by

$$(\mu * f)(x) = \mu(\tau_x \check{f}) = \int_{\mathbb{R}^n} (\tau_x \check{f})(y) d\mu(y) = \int_{\mathbb{R}^n} \check{f}(y-x) d\mu(y) = \int_{\mathbb{R}^n} f(x-y) d\mu(y).$$

Let  $\nu_t$  be the measure on  $\mathbb{R}^n$  with density  $k_t$ . We call  $\nu_t$  **Gaussian measure**. It satisfies

$$\nu_t * f(x) = \int_{\mathbb{R}^n} f(x-y) d\nu_t(y) = \int_{\mathbb{R}^n} f(x-y) k_t(y) dy = f * k_t(x), \quad x \in \mathbb{R}^n.$$