Hilbert-Schmidt operators and tensor products of Hilbert spaces

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Here I'm working through the notes *Hilbert-Schmidt operators, nuclear spaces, kernel theorem I* of Paul Garrett, University of Minnesota (it's easier for me just to give their title here rather than a link in case the link changes, so just Google it).

1 Tensor products of vector spaces and their completion

For k-vector spaces A, B let $\mathrm{Hom}_k(A, B)$ be the set of k-linear maps $A \to B$. One checks that $\mathrm{Hom}_k(A, B)$ is itself a k-vector space. For k-vector spaces A, B, C,

$$\mathrm{Hom}_k(A, \mathrm{Hom}_k(B, C)) \approx \mathrm{Hom}_k(A \otimes_k B, C),$$

where $\phi \in \mathrm{Hom}_k(A \to \mathrm{Hom}_k(B, C))$ is sent to the map $A \otimes_k B \to C$ that sends $a \otimes b$ to $\phi(a)b$. In particular with $C = k$,

$$\mathrm{Hom}_k(A, B^*) \approx (A \otimes_k B)^*.$$

If V, W are Hilbert spaces they are in particular vector spaces, and let $V \otimes_{\mathrm{alg}} W$ be their vector space tensor product. We define an inner product on $V \otimes_{\mathrm{alg}} W$ by defining it on the basis elements:

$$\langle x \otimes y, x' \otimes y' \rangle_{\mathrm{HS}} = \langle x, x' \rangle \langle y, y' \rangle.$$

Let $V \otimes_{\mathrm{HS}} W$ be the completion of $V \otimes_{\mathrm{alg}} W$ in the norm defined by this inner product. $V \otimes_{\mathrm{HS}} W$ is a Hilbert space; however, as Garrett shows it is not a categorical tensor product, and in fact if V and W are Hilbert spaces there is no Hilbert space that is their categorical tensor product. (We use the subscript HS because soon we will show that $V \otimes_{\mathrm{HS}} W$ is isomorphic as a Hilbert space to the Hilbert space of Hilbert-Schmidt operators $V \to W$.)

2 Finite-rank operators

If V, W are Hilbert spaces, let $F(V, W)$ be the set of bounded linear operators $V \to W$ whose range is finite dimensional. Define

$$\Phi : V^* \otimes_{\text{alg}} W \to F(V, W)$$

by

$$\Phi(\lambda \otimes w)(v) = (\lambda v)w.$$

Let $\{w_1, \ldots\}$ be an orthonormal basis of W. Let I be finite. Then,

$$\Phi\left(\sum_{i\in I} \lambda_i \otimes w_i\right)(v) = \sum_{i\in I} \Phi(\lambda_i \otimes w_i)(v) = \sum_{i\in I} (\lambda_i v)w_i \in \text{span}\{w_i : i \in I\}.$$

Thus Φ indeed sends an arbitrary element of $V^* \otimes_{\text{alg}} W$ to a bounded finite-rank operator $V \to W$. On the one hand, it is straightforward to check that Φ is injective (if it sends something to 0 that thing must be 0). On the other hand, let $T \in F(V, W)$. Since $T(V)$ is finite dimensional, it follows that $(\ker T)^\perp$ is finite dimensional. Let $v_1, \ldots, v_n$ be an orthonormal basis for $(\ker T)^\perp$. Define $\lambda_i \in V^*$ by $\lambda_i(v) = \langle v, v_i\rangle$. Then

$$\sum_{i=1}^{n} \lambda_i \otimes T v_i \in V^* \otimes_{\text{alg}} W,$$

and

$$\Phi\left(\sum_{i=1}^{n} \lambda_i \otimes T v_i\right) = T.$$

Thus Φ is surjective. So Φ is a linear isomorphism. Therefore it is equivalent for us to speak about bounded finite-rank operators $V \to W$ or about the vector space tensor product $V^* \otimes_{\text{alg}} W$. But when we talk about the completion of these spaces in the Hilbert-Schmidt norm, we can talk about the completion of $F(V, W)$ in $\text{Hom}(V, W)$, while we don't have a concrete space in which to talk about the completion of $V^* \otimes_{\text{alg}} W$.

3 Hilbert-Schmidt operators

We define an inner product on bounded finite-rank operators $V \to W$ using the inner product we have already defined on $V^* \otimes_{\text{alg}} W$ (and using the subscript HS for both): If $S, T \in F(V, W)$, define

$$\langle S, T\rangle_{\text{HS}} = \left\langle \Phi^{-1}(S), \Phi^{-1}(T)\right\rangle_{\text{HS}}.$$

Then, if $\{v_1, \ldots\}$ is an orthonormal basis for V and $\lambda_n(v) = \langle v, v_n\rangle$. When we showed that Φ is surjective we explicitly wrote the inverse of T under Φ and we use that here:

$$\begin{aligned}\langle S,T\rangle_{\mathrm{HS}} &= \left\langle \sum_n \lambda_n\otimes Sv_n, \sum_m \lambda_m \otimes Tv_m\right\rangle_{\mathrm{HS}}\\ &= \sum_n\sum_m \langle \lambda_n,\lambda_n\rangle\langle Sv_n, Tv_m\rangle\\ &= \sum_n \langle Sv_n, Tv_m\rangle.\end{aligned}$$

For $T\in F(V,W)$, we define $\|T\|_{\mathrm{HS}}^2 = \langle T,T\rangle_{\mathrm{HS}} = \sum_n |Tv_n|^2$.

We define $\mathrm{Hom}_{\mathrm{HS}}(V,W)$ to be the completion in $\mathrm{Hom}(V,W)$ of $F(V,W)$ in the above norm. $\mathrm{Hom}_{\mathrm{HS}}$ is a Hilbert space, since it's the completion of an inner product space.

If $S,T\in \mathrm{Hom}_{\mathrm{HS}}(V,W)$ and $\{v_1,\dots\}$ is an orthonormal basis of V, then

$$\langle S,T\rangle_{\mathrm{HS}} = \sum_n \langle Sv_n, Tv_n\rangle.$$

($\langle S,T\rangle_{\mathrm{HS}}$ was originally defined only for finite rank S and T, and thus the above is not merely by definition but something one checks.)

Let $\epsilon>0$ and let v_1 be a unit vector such that $|Tv_1|^2+\epsilon > \|T\|^2$, where $\|T\|$ is the operator norm of T. Then let $v_2,\dots$ be such that $v_1,v_2,\dots$ is an orthonormal basis of V. We have

$$\|T\|^2 < |Tv_1|^2+\epsilon \le \epsilon + \sum_n |Tv_n|^2 = \epsilon + \|T\|_{\mathrm{HS}}^2.$$

This is true for all $\epsilon>0$, so

$$\|T\|^2 \le \|T\|_{\mathrm{HS}}^2.$$

Since a limit of finite-rank operators in the operator norm is a compact operator, a Hilbert-Schmidt operator is compact (because a sequence of finite-rank operators that converges to an operator in the Hilbert-Schmidt norm will converge to it in the operator norm, and hence their limit will be a compact operator).

The adjoint of a Hilbert-Schmidt operator $V\to W$ is a Hilbert-Schmidt operator $W^*\to V^*$, i.e. $W\to V$ (Hilbert spaces).

Let $T: V\to W$ be a bounded linear operator and let $\{v_1,\dots\}$ be an orthonormal basis for V. If

$$\sum_n |Tv_n|^2 < \infty,$$

then T is a Hilbert-Schmidt operator, and this is the Hilbert-Schmidt norm of T. (This is a standard idea; if you want to show that a function is in L^2, then just determine its L^2 norm and see if this is finite.)

It is straightforward to show that if $S: W\to X$ is a bounded linear operator and $T: V\to W$ is a Hilbert-Schmidt operator, then $S\circ T: V\to X$ is a Hilbert-Schmidt operator. Likewise, using the fact that the adjoint of a Hilbert-Schmidt operator is Hilbert-Schmidt we can check that if $S: X\to V$ is a bounded linear operator then $T\circ S$ is a Hilbert-Schmidt operator.

4 Limits and tensor products

Let V, W, V_1, W_1 be Hilbert spaces and let $S : V_1 \to V$ and $T : W_1 \to W$ be Hilbert-Schmidt operators. Let $\beta : V \times W \to X$ be a bilinear map that is continuous (I don't say bounded here because β is not necessarily linear.) Let v_i be an orthonormal basis for V_1 and let w_j be an orthonormal basis for W_1. Define $B : V_1 \otimes_{\mathrm{HS}} W_1 \to X$ by

$$B\left(\sum_{i,j} c_{i,j} v_i \otimes w_j\right) = \sum_{i,j} c_{i,j} \beta(S v_i, T w_j).$$

This is a continuous bilinear map such that the composition $V_1 \times W_1 \to V_1 \otimes_{\mathrm{HS}} W_1 \to X$ is equal to the composition $V_1 \times W_1 \to V \times W \to X$. We can check that B is a Hilbert-Schmidt operator.

Let $V_0, V_1, \ldots$ be Hilbert spaces and let $\phi : V_i \to V_{i-1}$ be Hilbert-Schmidt operators. Let $V = \varprojlim_i V_i$, the projective limit in the category of topological vector spaces; so V has the coarsest topology such that the maps from it to each V_i are continuous. (One has actually to prove that there is a V that is the projective limit of the system of V_i and ϕ_i.) Garrett shows that if $V = \varprojlim_i V_i$ and $W = \varprojlim_j W_i$, then $\varprojlim_i V_w \otimes_{\mathrm{HS}} W_i$ is a categorical tensor product of V and W, in the category whose objects are Hilbert spaces and the above projective limits of Hilbert spaces.

5 Sobolev spaces

In this section I have referred to the notes *MAT218: Lecture Notes on Partial Differential Equations* of Steve Shkoller, UC Davis.

For $u \in L^1(\mathbb{T}^m)$,

$$\hat{u}(k) = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} e^{-ik\cdot x} u(x) dx, \qquad k \in \mathbb{Z}^m.$$

$u : \mathbb{T}^m \to \mathbb{C}$ and $\hat{u} : \mathbb{Z}^m \to \mathbb{C}$.

For $u, v \in L^2(\mathbb{T}^m)$,

$$\langle u, v\rangle_{L^2(\mathbb{T}^m)} = \left(\frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} u(x)\overline{v(x)} dx\right)^{1/2}$$

and

$$\langle \hat{u}, \hat{v}\rangle_{\ell^2(\mathbb{Z}^m)} = \sum_{k\in\mathbb{Z}^m} \hat{u}(k)\overline{\hat{v}(k)}.$$

Parseval's identity: $\|u\|_{L^2(\mathbb{T}^m)} = \|\hat{u}\|_{\ell^2(\mathbb{Z}^m)}$.

Let $\mathcal{D}'(\mathbb{T}^m) = (C^\infty(\mathbb{T}^m))'$, the dual space of $C^\infty(\mathbb{T}^m)$. These are the continuous linear functionals on $C^\infty(\mathbb{T}^m)$. We call elements of $\mathcal{D}'(\mathbb{T}^m)$ distributions. Distributions are not necessarily functions on $\mathbb{T}^m$, but their Fourier transforms are indeed functions on $\mathbb{Z}^m$.

Let $\langle k\rangle = \sqrt{1+|k|^2}$. For $s \in \mathbb{R}$, let

$$\langle u, v\rangle_{H^s(\mathbb{T}^m)} = \sum_{k\in\mathbb{Z}^m} \hat{u}(k)\overline{\hat{v}(k)} \langle k\rangle^{2s},$$

and let

$$H^s(\mathbb{T}^m) = \{u \in \mathcal{D}'(\mathbb{T}^m) : \|u\|^2_{H^s(\mathbb{T}^m)} < \infty\}.$$

These are called Sobolev spaces. I claim that $H^{-s}(\mathbb{T}^m) = (H^s(\mathbb{T}^m))'$. Let $\phi \in H^{-s}(\mathbb{T}^m)$. I'll show that ϕ does what an element of the dual space of $H^s(\mathbb{T}^m)$ is supposed to do. I am making no assumption about whether s is positive, 0, or negative here. For $u \in H^s(\mathbb{T}^m)$,

$$\begin{aligned} |\langle \phi, u\rangle| &= \left|\sum_{k\in\mathbb{Z}^m} \hat{\phi}(k)\overline{\hat{u}(k)}\right| \\ &= \left|\sum_{k\in\mathbb{Z}^m} \hat{\phi}(k)\hat{k}^{-s}\overline{\hat{u}(k)}\hat{k}^{s}\right| \\ &\leq \|\phi\|_{H^{-s}(\mathbb{T}^m)} \cdot \|u\|_{H^s(\mathbb{T}^m)}. \end{aligned}$$

Thus ϕ is a continuous linear functional on $H^s(\mathbb{T}^m)$.

Here we give an example of a function that belongs to certain Sobolev spaces but not others. Define H on $\mathbb{T}$ by

$$H(t) = \begin{cases} 1 & 0 \leq t < \pi, \\ 0 & \pi \leq t < 2\pi. \end{cases}$$

Then, for $k \neq 0$,

$$\hat{H}(k) = \frac{1}{2\pi}\int_0^{2\pi} H(t)e^{-ikt}dt = \frac{1}{2\pi}\int_0^{\pi} e^{-ikt}dt\frac{1}{2\pi}\cdot\frac{i}{k}(e^{-i\pi k}-1).$$

If k is odd, then $\hat{H}(k) = -\frac{i}{2\pi k}$. If $k \neq 0$ is even then $\hat{H}(k) = 0$. And $\hat{H}(0) = \frac{1}{2}$. Hence

$$\|H\|^2_{H^s(\mathbb{T})} = \sum_{k\in\mathbb{Z}} |\hat{H}(k)|^2 \langle k\rangle^{2s} = \frac{1}{4} + \frac{1}{4\pi^2}\sum_{k \text{ odd}} \frac{1}{k^2}\langle k\rangle^{2s}.$$

This is finite iff $2-2s > 1$, i.e. $s < \frac{1}{2}$. So if $s < \frac{1}{2}$ then $H \in H^s(\mathbb{T})$ and if $s \geq \frac{1}{2}$ then $H \notin H^s(\mathbb{T})$.

For s, define $\Lambda^s : \mathcal{D}'(\mathbb{T}^m) \to \mathcal{D}'(\mathbb{T}^m)$ by

$$(\Lambda^s u)(x) = \sum_{k\in\mathbb{Z}^m} |\hat{u}(k)| \langle k\rangle e^{ik\cdot x}.$$

(Think of this as defining the Fourier coefficients of a distribution.) We have

$$H^s(\mathbb{T}^m) = \Lambda^{-s}L^2(\mathbb{T}^m).$$

For any $r, s \in \mathbb{R}$,

$$\Lambda^s : H^r(\mathbb{T}^m) \to H^{r-s}(\mathbb{T}^m)$$

is an isomorphism of Hilbert spaces.

If $\epsilon > 0$, then I claim that $\Lambda^{-\epsilon} : H^s(\mathbb{T}^m) \to H^s(\mathbb{T}^m)$ is a compact operator. Define $\Lambda_N^{-\epsilon}$ by

$$(\Lambda_N^{-\epsilon} u)(x) = \sum_{|k| \leq N} |\hat{u}(k)| \langle k \rangle^{-\epsilon} e^{ik \cdot x}.$$

For each N, $\Lambda_N^{-\epsilon}$ is a finite-rank operator. And

$$\begin{aligned}
\left\| \Lambda^{-\epsilon} u - \Lambda_N^{-\epsilon} u \right\|_{H^s(\mathbb{T}^m} &= \left\| \sum_{|k| > N} |\hat{u}(k)| \langle k \rangle^{-\epsilon} e^{ik \cdot x} \right\|_{H^s(\mathbb{T}^m)} \\
&= \sum_{|k| > N} |\hat{u}(k)|^2 \langle k \rangle^{-2\epsilon} \langle k \rangle^{2s} \\
&< \langle N \rangle^{-2\epsilon} \|u\|^2_{H^s(\mathbb{T}^m} .
\end{aligned}$$

Thus $\Lambda_N^{-\epsilon} \to \Lambda^{-\epsilon}$ in the operator norm, so $\Lambda^{-\epsilon} : H^s(\mathbb{T}^m) \to H^s(\mathbb{T}^m)$ is a compact operator. It follows that the inclusion map $\iota : H^{s+\epsilon}(\mathbb{T}^m) \to H^s(\mathbb{T}^m)$ is compact. This is Rellich's theorem.

Fix s, and for $k \in \mathbb{Z}^m$ let

$$e_k = \frac{e^{ik \cdot x}}{\langle k \rangle^s}.$$

This is an orthonormal basis of $H^s(\mathbb{T}^m)$. For $t > 0$, I would like to know when the inclusion map $\iota : H^{s+t}(\mathbb{T}^m) \to H^s(\mathbb{T}^m)$ is a Hilbert-Schmidt operator. We have just seen that for all $t > 0$ it is a compact operator, and being Hilbert-Schmidt implies being compact. We have

$$\begin{aligned}
\sum_{k \in \mathbb{Z}^m} \|\iota(e_k)\|^2_{H^s(\mathbb{T}^m)} &= \sum_{k \in \mathbb{Z}^m} \left\| \frac{e^{ik \cdot x}}{\langle k \rangle^{s+t}} \right\|^2_{H^s(\mathbb{T}^m)} \\
&= \sum_{k \in \mathbb{Z}^m} \frac{1}{\langle k \rangle^{2s+2t}} \cdot \langle k \rangle^{2s} \\
&= \sum_{k \in \mathbb{Z}^m} \frac{1}{\langle k \rangle^{2t}}.
\end{aligned}$$

This is finite if and only if $2t > m$, i.e. $t > \frac{m}{2}$. That is, the inclusion map $H^{s+t}(\mathbb{T}^m) \to H^s(\mathbb{T}^m)$ is Hilbert-Schmidt if and only if $t > \frac{m}{2}$. In particular this gives us an example of a map that is compact but not Hilbert-Schmidt, because for all $t > 0$ the inclusion map is compact.

For all positive integers m, $m > \frac{m}{2}$. So for any s the inclusion map $H^{s+m}(\mathbb{T}^m) \to H^s(\mathbb{T}^m)$ is Hilbert-Schmidt. Thus we can take the projective limit of the Hilbert spaces $\ldots, H^{2m}(\mathbb{T}^m), H^m(\mathbb{T}^m), H^0(\mathbb{T}^m)$, with their inclusion maps. Call this limit $H^\infty(\mathbb{T}^m)$. Taking the projective limit of only the

Sobolev spaces $H^{jm}(\mathbb{T}^m)$ for $j \geq 0$, rather than of all $H^j(\mathbb{T}^m)$, is fair because the multiples of m are a cofinal set in the nonnegative integers: for every nonnegative integer there is a multiple of m that is larger than it. See Garrett's notes *Basic categorical constructions*.

Returning to Garrett's *Hilbert-Schmidt operators, nuclear spaces, kernel theorem I*, Garrett shows in them that

$$H^\infty(\mathbb{T}^m) \otimes H^\infty(\mathbb{T}^n) \approx H^\infty(\mathbb{T}^{m+n}).$$

The final fact that Garrett proves in his notes is the Schwartz kernel theorem for Sobolev spaces:

$$\mathrm{Hom}(H^\infty(\mathbb{T}^m), H^{-\infty}(\mathbb{T}^n)) \approx H^{-\infty}(\mathbb{T}^{m+n}).$$