

# Functions of bounded variation and a theorem of Khinchin

Jordan Bell

jordan.bell@gmail.com

Department of Mathematics, University of Toronto

March 12, 2016

For  $q \geq 1$  let

$$\mathcal{A}_q = \left\{ \frac{a}{q} : 0 \leq a \leq q, \gcd(a, q) = 1 \right\}.$$

The sets  $\mathcal{A}_q$  are pairwise disjoint. In particular  $0 \in \mathcal{A}_1$  and  $0 \notin \mathcal{A}_q$  for  $q > 1$ . We have

$$[0, 1] \cap \mathbb{Q} = \bigcup_{q \geq 1} \mathcal{A}_q.$$

Write  $\mu$  for Lebesgue measure on  $[0, 1]$ .

The following is a version of a theorem of Khinchin about continued fractions.<sup>1</sup> In the literature on Diophantine approximation it is usually proved with the Borel-Cantelli lemma, rather than the machinery of bounded variation and almost everywhere differentiability.

**Theorem 1** (Khinchin). *Let  $F : \mathbb{Z}_{\geq 1} \rightarrow \mathbb{R}_{>0}$  and let  $A$  be the set of those  $x \in [0, 1] \setminus \mathbb{Q}$  such that there are infinitely many  $q$  for which there is some  $\frac{a}{q} \in \mathcal{A}_q$  satisfying*

$$\left| \alpha - \frac{a}{q} \right| < \frac{1}{qF(q)}.$$

*If*

$$\sum_{q=1}^{\infty} \frac{1}{F(q)} < \infty,$$

*then  $\mu(A) = 0$ .*

*Proof.* Define  $f : [0, 1] \rightarrow \mathbb{R}_{>0}$  by

$$f(x) = \begin{cases} \frac{1}{qF(q)} & x \in \mathcal{A}_q \\ 0 & x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

---

<sup>1</sup>John J. Benedetto and Wojciech Czaja, *Integration and Modern Analysis*, p. 183, Theorem 4.3.3.

Let  $N \geq 1$ . If

$$0 = t_0 < t_1 < \dots < t_N = 1,$$

then

$$\begin{aligned} \sum_{j=0}^N f(t_j) &= \sum_{q=1}^{\infty} \sum_{j=0}^N f(t_j) \cdot 1_{\mathcal{A}_q}(t_j) \\ &= \sum_{q=1}^{\infty} \sum_{j=0}^N \frac{1}{qF(q)} \cdot 1_{\mathcal{A}_q}(t_j) \\ &\leq \sum_{q=1}^{\infty} \frac{1}{F(q)}, \end{aligned}$$

and so

$$\sum_{j=1}^N |f(t_j) - f(t_{j-1})| \leq 2 \sum_{j=1}^N f(t_j) \leq 2 \sum_{q=1}^{\infty} \frac{1}{F(q)}.$$

Therefore

$$V(f) \leq 2 \sum_{q=1}^{\infty} \frac{1}{F(q)} < \infty$$

by hypothesis, where  $V(f)$  denotes the variation of  $f$  on  $[0, 1]$ . The set  $D_f$  of points at which  $f$  is differentiable is a Borel set,<sup>2</sup> and because  $f$  has bounded variation,  $\mu(D_f) = 1$ .<sup>3</sup> Let  $E = D_f \setminus \mathbb{Q}$ , whose measure is  $\mu(E) = 1$ . Now let  $x \in E$ . There are  $x_n \in [0, 1] \setminus \mathbb{Q}$ ,  $x_n \neq x$ ,  $x_n \rightarrow x$ , with which

$$f'(x) = \lim_{n \rightarrow \infty} \frac{f(x_n) - f(x)}{x_n - x} = \lim_{n \rightarrow \infty} \frac{0 - 0}{x_n - x} = 0.$$

If  $\frac{a_n}{q_n} \rightarrow x$  with  $\frac{a_n}{q_n} \in \mathcal{A}_{q_n}$ , then

$$\frac{f(a_n/q_n) - f(x)}{\frac{a_n}{q_n} - x} = \frac{1}{q_n F(q_n) \left( \frac{a_n}{q_n} - x \right)} \rightarrow f'(x) = 0,$$

so  $q_n F(q_n) \left| \frac{a_n}{q_n} - x \right| \rightarrow \infty$ . There is thus some  $N$  such that if  $n \geq N$  then

$$q_n F(q_n) \left| \frac{a_n}{q_n} - x \right| \geq 1,$$

i.e. if  $n \geq N$  then

$$\left| x - \frac{a_n}{q_n} \right| \geq \frac{1}{q_n F(q_n)}.$$

<sup>2</sup>V. I. Bogachev, *Measure Theory*, volume 1, p. 371, Theorem 5.8.12.

<sup>3</sup>V. I. Bogachev, *Measure Theory*, volume 1, p. 335, Theorem 5.2.6.

Assume by contradiction that  $x \in A$ , so there are  $\frac{a_n}{q_n} \in \mathcal{A}_{q_n}$ ,  $\frac{a_n}{q_n} \neq \frac{a_m}{q_m}$  for  $n \neq m$ , with

$$\left| x - \frac{a_n}{q_n} \right| < \frac{1}{q_n F(q_n)},$$

and because  $\sum_{q=1}^{\infty} \frac{1}{F(q)} < \infty$  it holds that  $F(q) \rightarrow \infty$  and thus  $\frac{1}{q_n F(q_n)} \rightarrow 0$ . This means that  $x - \frac{a_n}{q_n} \rightarrow 0$ , which implies that  $x \notin E$ . We have shown that if  $x \in A$  then  $x \notin E$ , so  $A \subset [0, 1] \setminus E$  and hence  $\mu(A) \leq 1 - \mu(E) = 0$ .  $\square$