The Kolmogorov continuity theorem, Hölder continuity, and the Kolmogorov-Chentsov theorem

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1 Modifications

Let $(\Omega, \mathscr{F}, P)$ be a probability space, let I be a nonempty set, and let $(E, \mathscr{E})$ be a measurable space. A **stochastic process with index set** I **and state space** E is a family $(X_t)_{t\in I}$ of random variables $X_t : (\Omega, \mathscr{F}) \to (E, \mathscr{E})$. If X and Y are stochastic processes, we say that X **is a modification of** Y if for each $t \in I$,

$$P\{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\} = 1.$$

Lemma 1. *If X is a modification of Y, then X and Y have the same finite-dimensional distributions.*

Proof. For $t_1, \ldots, t_n \in I$, let $A_i \in \mathscr{E}$ for each $1 \le i \le n$, and let

$$A = \bigcap_{i=1}^{n} X_{t_i}^{-1}(A_i) \in \mathscr{F}, \qquad B = \bigcap_{i=1}^{n} Y_{t_i}^{-1}(A_i) \in \mathscr{F}.$$

If $\omega \in A \setminus B$ then there is some i for which $\omega \notin Y_{t_i}^{-1}(A_i)$, and $\omega \in X_{t_i}^{-1}(A_i)$ so $X_{t_i}(\omega) \neq Y_{t_i}(\omega)$. Therefore

$$A \triangle B \subset \bigcup_{i=1}^{n} \{\omega \in \Omega : X_{t_i}(\omega) \neq Y_{t_i}(\omega)\}.$$

Because X is a modification of Y, the right-hand side is a union of finitely many P-null sets, hence is itself a P-null set. A and B each belong to $\mathscr{F}$, so $P(A \triangle B) = 0$.[1] Because $P(A \triangle B) = 0$, $P(A) = P(B)$, i.e.

$$P(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = P(Y_{t_1} \in A_1, \ldots, Y_{t_n} \in A_n).$$

[1] We have not assumed that $(\Omega, \mathscr{F}, P)$ is a complete measure space, so we must verify that a set is measurable before speaking about its measure.

This implies that[2]

$$P_*(X_{t_1} \otimes \cdots \otimes X_{t_n}) = P_*(Y_{t_1} \otimes \cdots \otimes Y_{t_n}),$$

namely, X and Y have the same finite-dimensional distributions. $\square$

2 Continuous modifications

Let E be a Polish space with Borel σ-algebra $\mathscr{E}$. A stochastic process $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ is called **continuous** if for each $\omega \in \Omega$, the **path** $t \mapsto X_t(\omega)$ is continuous $\mathbb{R}_{\geq 0} \to E$.

A **dyadic rational** is an element of

$$D = \bigcup_{i=0}^{\infty} 2^{-i}\mathbb{Z}.$$

The **Kolmogorov continuity theorem** gives conditions under which a stochastic process whose state space is a Polish space has a continuous modification.[3] This is like the **Sobolev lemma**,[4] which states that if $f \in H^s(\mathbb{R}^d)$ and $s > k + \frac{d}{2}$, then there is some $\phi \in C^k(\mathbb{R}^d)$ such that $f = \phi$ almost everywhere. It does not make sense to say that an element of a Sobolev space is itself C^k, because elements of Sobolev spaces are equivalence classes of functions, but it does make sense to say that there is a C^k version of this element.

Theorem 2 (Kolmogorov continuity theorem)**.** *Suppose that $(\Omega, \mathscr{F}, P, (X_t)_{t\in\mathbb{R}_{\geq 0}})$ is a stochastic process with state space $\mathbb{R}^d$. If there are $\alpha, \beta, c > 0$ such that*

$$E(|X_t - X_s|^\alpha) \leq c|t-s|^{1+\beta}, \qquad s,t \in \mathbb{R}_{\geq 0}, \tag{1}$$

then the stochastic process has a continuous modification that itself satisfies (1)*.*

Proof. Let $0 < \gamma < \frac{\beta}{\alpha}$ and let

$$\delta = \beta - \alpha\gamma > 0.$$

For $m \geq 1$, let S_m be the set of all pairs (s,t) with

$$s,t \in \{j2^{-m} : 0 \leq j \leq 2^m\},$$

and $|s-t| = 2^{-m}$. There are $2 \cdot 2^m$ such pairs, i.e. $|S_m| = 2 \cdot 2^m$. Let

$$A_m = \bigcup_{(s,t)\in S_m} \{|X_s - X_t| \geq 2^{-\gamma m}\} \in \mathscr{F}.$$

[2] `http://individual.utoronto.ca/jordanbell/notes/finitedimdistributions.pdf`

[3] Heinz Bauer, *Probability Theory*, p. 335, Theorem 39.3. It was only after working through the proof given by Bauer that I realized that the statement is true when the state space is a Polish space rather than merely $\mathbb{R}^d$. In the proof I do not use that $|\cdot|$ is a norm on $\mathbb{R}^d$, and only use that $d(x,y) = |x-y|$ is a metric on $\mathbb{R}^d$, so it is straightforward to rewrite the proof.

[4] Walter Rudin, *Functional Analysis*, second ed., p. 202, Theorem 7.25.

For $(s,t) \in S_m$, using Chebyshev's inequality and (1) we get

$$\begin{aligned} P(|X_t - X_s| \geq 2^{-\gamma m}) &\leq (2^{\gamma m})^\alpha E(|X_t - X_s|^\alpha) \\ &\leq 2^{\alpha\gamma m} \cdot c|t-s|^{1+\beta} \\ &= c2^{\alpha\gamma m} 2^{-m(1+\beta)} \\ &< c2^{-m-\delta m}. \end{aligned}$$

Hence

$$P(A_m) \leq \sum_{(s,t)\in S_m} P\{|X_s - X_t| \geq 2^{-\gamma m}\} < \sum_{(s,t)\in S_m} c2^{-m-\delta m} = 2c \cdot 2^{-\delta m}.$$

Because $\sum_m P(A_m) < \infty$, the Borel-Cantelli lemma tells us that

$$P\left(\bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m\right) = P(N_0) = 0,$$

where for each $\omega \in \Omega \setminus N_0$ there is some $m_0(\omega)$ such that $\omega \notin A_m$ when $m \geq m_0(\omega)$. That is, for $\omega \in \Omega \setminus N_0$ there is some $m_0(\omega)$ such that

$$|X_t(\omega) - X_s(\omega)| < 2^{-\gamma m}, \qquad m \geq m_0(\omega), \quad (s,t) \in S_m. \tag{2}$$

Now let $\omega \in \Omega \setminus N_0$ and let $s,t \in [0,1]$ be dyadic rationals satisfying

$$0 < |s-t| \leq 2^{-m_0(\omega)}.$$

Let $m = m(s,t)$ be the greatest integer such that $|s-t| \leq 2^{-m}$:

$$2^{-m-1} < |s-t| \leq 2^{-m}, \tag{3}$$

which implies that $m \geq m_0(\omega)$. There are some $i_0, j_0 \in \{0,1,2,3,\dots,2^m\}$ such that

$$s_0 = i_0 2^{-m} \leq s < (i_0+1)2^{-m}, \qquad t_0 = j_0 2^{-m} \leq t < (j_0+1)2^{-m}.$$

As $0 \leq s - s_0 < 2^{-m}$ and $0 \leq t - t_0 < 2^{-m}$, there are sequences $\sigma_j, \tau_j \in \{0,1\}$, $j > m$, each of which have cofinitely many zero entries, such that

$$s = s_0 + \sum_{j>m} \sigma_j 2^{-j}, \qquad t = t_0 + \sum_{j>m} \tau_j 2^{-j}.$$

Because $0 \leq s - s_0 < 2^{-m}$ and $\leq t - t_0 < 2^{-m}$,

$$2^{-m} > |(s-s_0) - (t-t_0)| = |(s-t) - (s_0 - t_0)| \geq |s_0 - t_0| - |s-t|,$$

and with (3),

$$|s_0 - t_0| < 2^{-m} + |s-t| \leq 2^{-m} + 2^{-m} = 2^{-m+1}.$$

Thus $|i_0 - j_0| < 2$, so $|i_0 - j_0| \in \{0, 1\}$ and so either $s_0 = t_0$ or $(s_0, t_0) \in S_m$. In the first case, $|X_{t_0}(\omega) - X_{s_0}(\omega)| = 0$. In the second case, since $m \geq m_0(\omega)$, by (2) we have

$$|X_{t_0}(\omega) - X_{s_0}(\omega)| < 2^{-\gamma m}. \tag{4}$$

Define by induction

$$s_n = s_{n-1} + \sigma_{m+n} 2^{-(m+n)}, \qquad n \geq 1,$$

i.e.

$$s_n = s_0 + \sum_{m<j\leq m+n} \sigma_j 2^{-j}.$$

For each $n \geq 1$, $s_n - s_{n-1} \in \{0, 2^{-(m+n)}\}$, so either $s_n = s_{n-1}$ or $(s_{n-1}, s_n) \in S_{m+n}$, and because $m + n \geq m + 1 > m_0(\omega)$, applying (2) yields

$$|X_{s_n}(\omega) - X_{s_{n-1}}(\omega)| < 2^{-\gamma(m+n)}.$$

Because the sequence σ_j is eventually equal to 0, the sequence s_n is eventually equal to s. Thus

$$\sum_{n=1}^{\infty} (X_{s_n}(\omega) - X_{s_{n-1}}(\omega)) = X_s(\omega) - X_{s_0}(\omega),$$

whence

$$|X_s(\omega) - X_{s_0}(\omega)| \leq \sum_{n=1}^{\infty} |X_{s_n}(\omega) - X_{s_{n-1}}(\omega)| < \sum_{n=1}^{\infty} 2^{-\gamma(m+n)} = \frac{2^{-\gamma(m+1)}}{1 - 2^{-\gamma}}.$$

By the same reasoning we get

$$|X_t(\omega) - X_{t_0}(\omega)| < \frac{2^{-\gamma(m+1)}}{1 - 2^{-\gamma}}.$$

Using these and (4) yields

$$\begin{aligned}
|X_t(\omega) - X_s(\omega)| &\leq |X_t(\omega) - X_{t_0}(\omega)| + |X_{t_0}(\omega) - X_{s_0}(\omega)| + |X_s(\omega) - X_{s_0}(\omega)| \\
&< \frac{2^{-\gamma(m+1)}}{1 - 2^{-\gamma}} + 2^{-\gamma m} + \frac{2^{-\gamma(m+1)}}{1 - 2^{-\gamma}} \\
&= C \cdot 2^{-\gamma(m+1)},
\end{aligned}$$

for $C = 2^\gamma + \frac{2}{1-2^{-\gamma}}$. By (3), $2^{-(m+1)} < |t - s|$, hence

$$|X_t(\omega) - X_s(\omega)| \leq C|t - s|^\gamma. \tag{5}$$

This is true for all dyadic rationals $s, t \in [0, 1]$ with $|s - t| \leq 2^{-m_0(\omega)}$; when $|s - t| = 0$ it is immediate.

For $k \geq 1$, let $X_t^k = X_{k+t}$, which satisfies (1). By what we have worked out, there is a P-null set $N_1' \in \mathscr{F}$ such that for each $\omega \in \Omega \setminus N_1'$ there is some

$m_1'(\omega)$ such that $m \geq m_1'(\omega)$ and $(s,t) \in S_m$ imply that $|X_t^1(\omega) - X_s^1(\omega)| < 2^{-\gamma m}$. Let $N_1 = N_0 \cup N_1'$, which is P-null, and for $\omega \in \Omega \setminus N_1$ let $m_1(\omega) = \max\{m_0(\omega), m_1'(\omega)\}$. For $s,t \in D \cap [0,1]$ with $|s-t| \leq 2^{-m_1(\omega)}$, what we have worked out yields

$$|X_t(\omega) - X_s(\omega)| \leq C|t-s|^\gamma, \qquad |X_t^1(\omega) - X_s^1(\omega)| \leq C|t-s|^\gamma.$$

By induction, we get that for each $k \geq 1$ there are P-null sets $N_0 \subset N_1 \subset \cdots \subset N_k$ and for each $\omega \in \Omega \setminus N_k$ there is some $m_k(\omega)$ such that for $s,t \in D \cap [0,1]$ with $|s-t| \leq 2^{-m_k(\omega)}$,

$$\begin{aligned} |X_t(\omega) - X_s(\omega)| &\leq C|t-s|^\gamma \\ |X_t^1(\omega) - X_s^1(\omega)| &\leq C|t-s|^\gamma \\ &\cdots \\ |X_t^k(\omega) - X_s^k(\omega)| &\leq C|t-s|^\gamma. \end{aligned}$$

Let

$$N_\gamma = \bigcup_{k \geq 1} N_k,$$

which is an increasing sequence of sets whose union is P-null. For $\omega \in \Omega \setminus N_\gamma$, there is a nondecreasing sequence $m_k(\omega)$ such that when $0 \leq j \leq k$ and $s,t \in D \cap [j,j+1]$ with $|s-t| \leq 2^{-m_k(\omega)}$, it is the case that $|X_t(\omega) - X_s(\omega)| \leq C|t-s|^\gamma$. For $s,t \in D \cap [0,k+1]$ with $|s-t| \leq 2^{-m_k(\omega)}$, because $|s-t| \leq \frac{1}{2}$, either there is some $0 \leq j \leq k$ for which $s,t \in [j,j+1]$ or there is some $1 \leq j \leq k$ for which, say, $s < j < t$. In the first case, $|X_t(\omega) - X_s(\omega)| \leq C|t-s|^\gamma$. In the second case, because $|j-s| < |t-s| \leq 2^{-m_k(\omega)}$ and $|t-j| < |t-s| \leq 2^{-m_k(\omega)}$, we get, because $s,j \in D \cap [j-1,j]$ and $j,t \in D \cap [j,j+1]$,

$$\begin{aligned} |X_t(\omega) - X_s(\omega)| &\leq |X_t(\omega) - X_j(\omega)| + |X_j(\omega) - X_s(\omega)| \\ &\leq C|t-j|^\gamma + C|j-s|^\gamma \\ &< 2C|t-s|^\gamma. \end{aligned}$$

Thus for

$$C_\gamma = 2C = 2^{\gamma+1} + \frac{4}{1-2^{-\gamma}},$$

we have established that for $\omega \in \Omega \setminus N_\gamma$, $k \geq 1$, and $s,t \in D \cap [0,k+1]$ satisfying $|t-s| \leq 2^{-m_k(\omega)}$, it is the case that

$$|X_t(\omega) - X_s(\omega)| \leq C_\gamma |t-s|^\gamma. \tag{6}$$

This implies that for each $\omega \in \Omega \setminus N_\gamma$ and for $k \geq 1$, the mapping $t \mapsto X_t(\omega)$ is uniformly continuous on $D \cap [0,k+1]$. For $t \in \mathbb{R}_{\geq 0}$ and $\omega \in \Omega \setminus N_\gamma$, define

$$Y_t(\omega) = \lim_{\substack{s \to t \\ s \in D}} X_s(\omega). \tag{7}$$

For each $k \geq 0$, because $t \mapsto X_t(\omega)$ is uniformly continuous $D \cap [0, k+1] \to \mathbb{R}^d$, where $D \cap [0, k+1]$ is dense in $[0, k+1]$ and $\mathbb{R}^d$ is a complete metric space, the map $t \mapsto Y_t(\omega)$ is uniformly continuous $[0, k+1] \to \mathbb{R}^d$.[5] Then $t \mapsto Y_t(\omega)$ is continuous $\mathbb{R}_{\geq 0} \to \mathbb{R}^d$. For $\omega \in N_\gamma$, we define

$$Y_t(\omega) = 0, \qquad t \in \mathbb{R}_{\geq 0}.$$

Then for each $\omega \in \Omega$, $t \mapsto Y_t(\omega)$ is continuous $\mathbb{R}_{\geq 0} \to \mathbb{R}^d$. For $t \in \mathbb{R}_{\geq 0}$, $\omega \mapsto Y_t(\omega)$ is the pointwise limit of the sequence of mappings $\omega \mapsto X_s(\omega)$ as $s \to t$, $s \in D$. For each $s \in D$, $\omega \mapsto X_s(\omega)$ is measurable $\mathscr{F} \to \mathscr{B}_{\mathbb{R}^d}$, which implies that $\omega \mapsto Y_t(\omega)$ is itself measurable $\mathscr{F} \to \mathscr{B}_{\mathbb{R}^d}$.[6] Namely, $(Y_t)_{t \in \mathbb{R}_{\geq 0}}$ is a continuous stochastic process.

We must show that Y is a modification of X. For $s \in D$, for all $\omega \in \Omega \setminus N_\gamma$ we have $Y_s(\omega) = X_s(\omega)$. For $t \in \mathbb{R}_{\geq 0}$, there is a sequence $s_n \in D$ tending to t, and then for all $\omega \in \Omega \setminus N_\gamma$ by (7) we have $X_{s_n}(\omega) \to Y_t(\omega)$. $P(N_\gamma) = 0$, namely, X_{s_n} converges to Y_t almost surely. Because X_{s_n} converges to Y_t almost surely and P is a probability measure, X_{s_n} converges in measure to Y_t.[7] On the other hand, for $\eta > 0$, by Chebyshev's inequality and (1),

$$P\{|X_{s_n} - X_t| \geq \eta\} \leq \eta^{-\alpha} E(|X_{s_n} - X_t|^\alpha) \leq \eta^{-\alpha} \cdot c|s_n - t|^{1+\beta},$$

and because this is true for each $\eta > 0$, this shows that X_{s_n} converges in measure to X_t. Hence, the limits Y_t and X_t are equal as equivalence classes of measurable functions $\Omega \to \mathbb{R}^d$.[8] That is, $P\{Y_t = X_t\} = 1$. This is true for each $t \in \mathbb{R}_{\geq 0}$, showing that Y is a modification of X, completing the proof. $\Box$

3 Hölder continuity

Let (X, d) and (Y, ρ) be metric spaces, let $0 < \gamma < 1$, and let $\phi : X \to Y$ be a function. For $x_0 \in X$, we say that ϕ is **γ-Hölder continuous at** x_0 if there is some $0 < \epsilon_{x_0} < 1$ and some C_{x_0} such that when $d(x, x_0) < \epsilon_{x_0}$,

$$\rho(\phi(x), \phi(x_0)) \leq C_{x_0} d(x, x_0)^\gamma.$$

We say that ϕ is **locally γ-Hölder continuous** if for each $x_0 \in X$ there is some $0 < \epsilon_{x_0} < 1$ and some C_{x_0} such that when $d(x, x_0) < \epsilon_{x_0}$ and $d(y, x_0) < \epsilon_{x_0}$,

$$\rho(\phi(x), \phi(y)) \leq C_{x_0} d(x, y)^\gamma.$$

We say that ϕ is **uniformly γ-Hölder continuous** if there is some C such that for all $x, y \in X$,

$$\rho(\phi(x), \phi(y)) \leq C d(x, y)^\gamma.$$

[5] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 77, Lemma 3.11.

[6] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 142, Lemma 4.29.

[7] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 479, Theorem 13.37.

[8] `http://individual.utoronto.ca/jordanbell/notes/L0.pdf`

We establish properties of Hölder continuous functions in the following.[9]

Lemma 3. *Let V be a nonempty subset of $\mathbb{R}_{\geq 0}$, let $0 < \gamma < 1$, and let $f : V \to \mathbb{R}^d$ be locally γ-Hölder continuous.*

1. *If $0 < \gamma' < \gamma$ then f is locally γ'-Hölder continuous.*
2. *If V is compact, then f is uniformly γ-Hölder continuous.*
3. *If V is an interval of length $T > 0$ and there is some $\epsilon > 0$ and some C such that for all $s, t \in V$ with $|t - s| \leq \epsilon$ we have*

$$|f(t) - f(s)| \leq C|t - s|^\gamma, \tag{8}$$

then

$$|f(t) - f(s)| \leq C \left\lceil \frac{T}{\epsilon} \right\rceil^{1-\gamma} |t - s|^\gamma, \qquad s, t \in V.$$

Proof. For $t_0 \in \mathbb{R}_{\geq 0}$, there is some $0 < \epsilon_{t_0} < 1$ and some C_{t_0} such that when $|t - t_0| < \epsilon_{t_0}$,

$$|f(t) - f(t_0)| \leq C_{t_0}|t - t_0|^\gamma \leq C_{t_0}|t - t_0|^{\gamma'},$$

showing that f is locally γ'-Hölder continuous.

With the metric inherited from $\mathbb{R}_{\geq 0}$, V is a compact metric space. For $t \in V$ and $\epsilon > 0$, write

$$B_\epsilon(t) = \{v \in V : |v - t| < \epsilon\},$$

which is an open subset of V. Because f is locally γ-Hölder continuous, for each $t \in V$ there is some $0 < \epsilon_t < 1$ and some C_t such that for all $u, v \in B_{\epsilon_t}(t)$,

$$|f(u) - f(v)| \leq C_t|u - v|^\gamma. \tag{9}$$

Write $U_t = B_{\epsilon_t}(t)$. Because $t \in U_t$, $\{U_t : t \in V\}$ is an open cover of V, and because V is compact there are $t_1, \ldots, t_n \in V$ such that $\mathfrak{U} = \{U_{t_1}, \ldots, U_{t_n}\}$ is an open cover of V. Because V is a compact metric space, there is a **Lebesgue number** $\delta > 0$ of the open cover $\mathfrak{U}$:[10] for each $t \in V$, there is some $1 \leq i \leq n$ such that $B_\delta(t) \subset U_{t_i}$. Let

$$C = \max\{C_{t_1}, \ldots, C_{t_n}, 2\|f\|_u \delta^{-\gamma}\},$$

For $s, t \in V$ with $|t - s| < \delta$, i.e. $s \in B_\delta(t)$, there is some $1 \leq i \leq n$ with $s, t \in U_{t_i}$. By (9),

$$|f(s) - f(t)| \leq C_{t_i}|s - t|^\gamma \leq C|s - t|^\gamma.$$

On the other hand, for $s, t \in V$ with $|t - s| \geq \delta$,

$$|f(s) - f(t)| \leq 2\|f\|_u \leq 2\|f\|_u \left(\frac{|s - t|}{\delta}\right)^\gamma = 2\|f\|_u \delta^{-\gamma}|s - t|^\gamma \leq C|s - t|^\gamma.$$

[9] Achim Klenke, *Probability Theory: A Comprehensive Course*, p. 448, Lemma 21.3.

[10] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 85, Lemma 3.27.

Thus, for all $s, t \in V$,

$$|f(s) - f(t)| \le C|s-t|^\gamma,$$

showing that f is uniformly γ-Hölder continuous.

Let $n = \left\lceil \frac{T}{\epsilon} \right\rceil$. For $s, t \in V$, because V is an interval of length T, $|s-t| \le T \le \epsilon n$, and then applying (8), because $\frac{|t-s|}{n} \le \epsilon$,

$$\begin{aligned}
|f(t) - f(s)| &= \left| \sum_{k=1}^n f\left(s + (t-s)\frac{k}{n}\right) - f\left(s + (t-s)\frac{k-1}{n}\right) \right| \\
&\le \sum_{k=1}^n \left| f\left(s + (t-s)\frac{k}{n}\right) - f\left(s + (t-s)\frac{k-1}{n}\right) \right| \\
&\le \sum_{k=1}^n C \left| \frac{t-s}{n} \right|^\gamma \\
&= C n^{1-\gamma} |t-s|^\gamma.
\end{aligned}$$

□

The following theorem does not speak about a version of a stochastic process. Rather, it shows what can be said about a stochastic process that satisfies (1) when almost all of its sample paths are continuous.[11]

Theorem 4. *If a stochastic process $(X_t)_{t \in \mathbb{R}_{\ge 0}}$ with state space $\mathbb{R}^d$ satisfies* (1) *and for almost every $\omega \in \Omega$ the map $t \mapsto X_t(\omega)$ is continuous $\mathbb{R}_{\ge 0} \to \mathbb{R}^d$, then for almost every $\omega \in \Omega$, for every $0 < \gamma < \frac{\beta}{\alpha}$, the map $t \mapsto X_t(\omega)$ is locally γ-Hölder continuous.*

Proof. There is a P-null set $N \in \mathscr{F}$ such that for $\omega \in \Omega \setminus N$, the map $t \mapsto X_t(\omega)$ is continuous $\mathbb{R}_{\ge 0} \to \mathbb{R}^d$. For each $0 < \gamma < \frac{\beta}{\alpha}$, we have established in (6) that there is a P-null set $N_\gamma \in \mathscr{F}$ such that for $k \ge 1$ there is some $m_k(\omega)$ such that when $s, t \in D \cap [0, k+1]$ and $|t-s| \le 2^{-m_k(\omega)}$,

$$|X_t(\omega) - X_s(\omega)| \le C_\gamma |t-s|^\gamma, \tag{10}$$

where $C_\gamma = 2^{\gamma+1} + \frac{4}{1-2^{-\gamma}}$. Write $\delta(k, \omega) = 2^{-m_k(\omega)}$, and let $M_\gamma = N_\gamma \cup N$. For $\omega \in \Omega \setminus M_\gamma$, the map $t \mapsto X_t(\omega)$ is continuous $\mathbb{R}_{\ge 0} \to \mathbb{R}^d$. For $k \ge 1$ and for $s, t \in [0, k+1]$ satisfying $|s-t| \le \delta(k, \omega)$, say with $s \le t$, let $m = \frac{t-s}{2}$ and let $s \le s_n \le t$ be a sequence of dyadic rationals decreasing to s and let $s \le t_n \le t$ be a sequence of dyadic rationals inceasing to t. Then $s_n, t_n \in D \cap [0, k+1]$ and $|s_n - t_n| \le |s-t| \le \delta(k, \omega)$, so by (10),

$$|X_{t_n}(\omega) - X_{s_n}(\omega)| \le C_\gamma |t_n - s_n|^\gamma.$$

[11] Heinz Bauer, *Probability Theory*, p. 338, Theorem 39.4.

Because $\omega \in \Omega \setminus N$, $X_{t_n}(\omega) \to X_t(\omega)$ and $X_{s_n}(\omega) \to X_s(\omega)$, so

$$\begin{aligned} |X_t(\omega) - X_s(\omega)| &\leq |X_t(\omega) - X_{t_n}(\omega)| + |X_{t_n}(\omega) - X_{s_n}(\omega)| + |X_s(\omega) - X_{s_n}(\omega)| \\ &\leq |X_t(\omega) - X_{t_n}(\omega)| + C_\gamma |t_n - s_n|^\gamma + |X_s(\omega) - X_{s_n}(\omega)| \\ &\downarrow C_\gamma |t - s|^\gamma, \end{aligned}$$

thus

$$|X_t(\omega) - X_s(\omega)| \leq C_\gamma |t - s|^\gamma,$$

showing that for $0 < \gamma < \frac{\beta}{\alpha}$ and $\omega \in \Omega \setminus M_\gamma$, the map $t \mapsto X_t(\omega)$ is locally γ-Hölder continuous.

Let $0 < \gamma_n < \frac{\beta}{\alpha}$ be a sequence increasing to $\frac{\beta}{\alpha}$ and let

$$M = \bigcup_{n \geq 1} M_{\gamma_n},$$

which is a P-null set. Let $0 < \gamma < \frac{\beta}{\alpha}$ and let n be such that $\gamma_n \geq \gamma$. For $\omega \in \Omega \setminus M$, the map $t \mapsto X_t(\omega)$ is locally γ_n-Hölder continuous, and because $\gamma \leq \gamma_n$ this implies that the map is locally γ-Hölder continuous, completing the proof. □

Bauer attributes the following theorem to Kolgmorov and Chentsov.[12] It does not merely state that for any $0 < \gamma < \frac{\beta}{\alpha}$ there is a modification that is locally γ-Hölder continuous, but that there is a modification that for all $0 < \gamma < \frac{\beta}{\alpha}$ is locally γ-Hölder continuous.[13]

Theorem 5 (Kolmogorov-Chentsov theorem)**.** *If a stochastic process $(X_t)_{t \in \mathbb{R}_{\geq 0}}$ with state space $\mathbb{R}^d$ satisfies* (1), *then X has a modification Y such that for all $\omega \in \Omega$ and $0 < \gamma < \frac{\beta}{\alpha}$, the path $t \mapsto Y_t(\omega)$ is locally γ-Hölder continuous.*

Proof. Applying the Kolmogorov continuity theorem, there is a continuous modification Z of X that also satisfies (1). By Theorem 4, there is a P-null set M such that for $\omega \in \Omega \setminus M$ and $0 < \gamma < \frac{\beta}{\alpha}$, the map $t \mapsto Z_t(\omega)$ is locally γ-Hölder continuous. For $t \in \mathbb{R}_{\geq 0}$, define

$$Y_t(\omega) = \begin{cases} Z_t(\omega) & \omega \in \Omega \setminus M \\ 0 & \omega \in M, \end{cases}$$

i.e. $Y_t = 1_{\Omega \setminus M} Z_t$, which is measurable $\mathscr{F} \to \mathscr{B}_{\mathbb{R}^d}$, and so $(Y_t)_{t \in \mathbb{R}_{\geq 0}}$ is a stochastic process. For every $\omega \in \Omega$ and $0 < \gamma < \frac{\beta}{\alpha}$, the map $t \mapsto Y_t(\omega)$ is locally γ-Hölder continuous. For $t \in \mathbb{R}_{\geq 0}$,

$$\{X_t \neq Y_t\} = \{X_t \neq Y_t, X_t = Z_t\} \cup \{X_t \neq Y_t, X_t \neq Z_t\} \subset \{Y_t \neq Z_t\} \cup \{X_t \neq Z_t\}.$$

Because $P(Y_t \neq Z_t) = P(M) = 0$ and $P(X_t \neq Z_t) = 0$, since Z is a modification of X, we get $P(X_t \neq Y_t) = 0$, namely, Y is a modification of X. □

[12] Nikolai Nikolaevich Chentsov, 1930–1993, obituary in Russian Math. Surveys **48** (1993), no. 2, 161–166.

[13] Heinz Bauer, *Probability Theory*, p. 339, Corollary 39.5.