Lévy's inequality, Rademacher sums, and Kahane's inequality

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1 Lévy's inequality

Let $(\Omega, \mathscr{A}, P)$ be a probability space. A **random variable** is a Borel measurable function $\Omega \to \mathbb{R}$. For a random variable X, we denote by X_*P the pushforward measure of P by X. X_*P is a Borel probability measure on $\mathbb{R}$, called the **distribution of** X. A random variable X is called **symmetric** when the distribution of X is equal to the distribution of $-X$. Because the collection $\{(-\infty, a] : a \in \mathbb{R}\}$ generates the Borel σ-algebra of $\mathbb{R}$, the statement that $X_*P = (-X)_*P$ is equivalent to the statement that for all $a \in \mathbb{R}$,

$$P(\{\omega \in \Omega : X(\omega) \leq a\}) = P(\omega \in \Omega : -X(\omega) \leq a\}).$$

The following is **Lévy's inequality**.[1]

Theorem 1 (Lévy's inequality). *Suppose that χ_k, $k \geq 1$, are independent symmetric random variables, that U is a real or complex Banach space, and that $u_k \in U$, $k \geq 1$. Then for each $a > 0$ and for each $n \geq 1$,*

$$P\left(\max_{1 \leq k \leq n} \left\| \sum_{1 \leq j \leq k} \chi_j u_j \right\| \geq a\right) \leq 2 \cdot P\left(\left\| \sum_{1 \leq j \leq n} \chi_j u_j \right\| \geq a\right).$$

Proof. Let $S_0 = 0$ and for $1 \leq k \leq n$,

$$S_k(\omega) = \sum_{j=1}^{k} \chi_j(\omega) u_j, \qquad \omega \in \Omega.$$

[1] Joe Diestel, Hans Jarchow, and Andrew Tonge, *Absolutely Summing Operators*, p. 213, Theorem 11.3.

For $1 \leq k \leq n$, the function $\omega \mapsto (\chi_1(\omega), \ldots, \chi_k(\omega))$ is Borel measurable $\Omega \to \mathbb{R}^k$.[2] The function $(t_1, \ldots, t_k) \mapsto \sum_{j=1}^k t_j u_j$ is continuous $\mathbb{R}^k \to U$. And the function $u \mapsto \|u\|$ is continuous $U \to \mathbb{R}$. Therefore $\omega \mapsto \|S_k(\omega)\|$, the composition of these functions, is Borel measurable $\Omega \to \mathbb{R}$. This then implies that $\omega \mapsto \max_{1 \leq k \leq n} \|S_k(\omega)\|$ is Borel measurable $\Omega \to \mathbb{R}$. Let

$$A = \{\omega \in \Omega : \max_{1 \leq k \leq n} \|S_k(\omega)\| \geq a\}, \quad B = \{\omega \in \Omega : \|S_n(\omega)\| \geq a\},$$

for which $B \subset A$. For $1 \leq k \leq n$, let

$$A_k = \bigcap_{0 \leq j < k} \{\omega \in \Omega : \|S_j(\omega)\| < a \text{ and } \|S_k(\omega)\| \geq a\}.$$

It is apparent that that $A_1, \ldots, A_n$ are pairwise disjoint and that $A = \bigcup_{k=1}^n A_k$.

For $1 \leq k \leq n$, let

$$T_{n,k}(\omega) = S_k(\omega) - \sum_{j=k+1}^n \chi_j(\omega) u_j = \sum_{j=1}^k \chi_j(\omega) u_j - \sum_{j=k+1}^n \chi_j(\omega) u_j, \qquad \omega \in \Omega,$$

in other words, $S_n + T_{n,k} = 2S_k$. Let

$$U_k = A_k \cap B, \qquad V_k = A_k \cap \{\omega \in \Omega : \|T_{n,k}(\omega)\| \geq a\}.$$

If $\omega \in A_k$, then

$$\|S_n(\omega) + T_{n,k}(\omega)\| = 2\|S_k(\omega)\| \geq 2a,$$

which implies that at least one of the inequalities $\|S_n(\omega)\| \geq a$ or $\|T_{n,k}(\omega)\| \geq a$ is true. Therefore

$$A_k = U_k \cup V_k.$$

Because $\chi_1, \ldots, \chi_n$ are independent, the random vector $X = (\chi_1, \ldots, \chi_n) : \Omega \to \mathbb{R}^n$ has the pushforward measure

$$X_* P = \chi_{1*} P \times \cdots \times \chi_{n*} P,$$

and for each $1 \leq k \leq n$, the random vector $X_k = (\chi_1, \ldots, \chi_k, -\chi_{k+1}, \ldots, -\chi_n) : \Omega \to \mathbb{R}^n$ has the pushforward measure

$$X_{k*} P = \chi_{1*} P \times \cdots \chi_{k*} P \times (-\chi_{k+1})_* P \times \cdots (-\chi_n)_* P,$$

and because each χ_j is symmetric, these pushforward measures are equal. Define $\sigma_k : \mathbb{R}^k \to \mathbb{R}$ by

$$\sigma_k(t_1, \ldots, t_k) = \left\| \sum_{j=1}^k t_j u_j \right\|, \qquad (t_1, \ldots, t_k) \in \mathbb{R}^k,$$

[2] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhikers Guide*, third ed., p. 152, Lemma 4.49.

define $\sigma_0 = 0$, and set

$$H_k = \left(\bigcap_{0 \le j < k} \{(t_1, \dots, t_n) \in \mathbb{R}^n : \sigma_j(t_1, \dots, t_j) < a\} \right) \\ \cap \{(t_1, \dots, t_n) \in \mathbb{R}^n : \sigma_k(t_1, \dots, t_k) \ge a, \sigma_n(t_1, \dots, t_n) \ge a\}.$$

Because each σ_j is continuous, H_k is a Borel set in $\mathbb{R}^n$. Then we have

$$\begin{aligned} P(U_k) &= P(A_k \cap B) \\ &= P(X^{-1}(H_k)) \\ &= (X_* P)(H_k) \\ &= (X_{k*} P)(H_k) \\ &= P(X_k^{-1}(H_k)) \\ &= P(A_k \cap \{\omega \in \Omega : \|T_{n,k}(\omega)\| \ge a\}) \\ &= P(V_k); \end{aligned}$$

among the above equalities, the two equalities that deserve chewing on are

$$P(A_k \cap B) = P(X^{-1}(H_k)) \quad \text{and} \quad P(X_k^{-1}(H_k)) = P(A_k \cap \{\omega \in \Omega : \|T_{n,k}(\omega)\| \ge a\}).$$

Thus we have

$$P(A_k) = P(U_k \cup V_k) \le P(U_k) + P(V_k) = 2P(U_k) = 2P(A_k \cap B).$$

Therefore

$$\begin{aligned} P(A) &= \sum_{k=1}^n P(A_k) \\ &\le \sum_{k=1}^n 2P(A_k \cap B) \\ &= 2P(A \cap B) \\ &= 2P(B), \end{aligned}$$

proving the claim. □

2 Rademacher sums

Suppose that $\epsilon_n : (\Omega, \mathscr{A}, P) \to (\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \lambda)$, $n \ge 1$, are independent random variables each with the **Rademacher distribution**: for each n,

$$\epsilon_{n*} P = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1,$$

in other words, $P(\epsilon_n = 1) = \frac{1}{2}$ and $P(\epsilon_n = -1) = \frac{1}{2}$.

We now use Lévy's inequality to prove the following for independent random variables with the Rademacher distribution.[3]

Theorem 2. *Suppose that X is a real or complex Banach space, and that $x_k \in X$, $k \geq 1$. Then for each $a > 0$ and for each $n \geq 1$,*

$$P\left(\left\|\sum_{k=1}^n \epsilon_k x_k\right\| \geq 2a\right) \leq 4\left(P\left(\left\|\sum_{k=1}^n \epsilon_k x_k\right\| \geq a\right)\right)^2.$$

Proof. Let $S_0 = 0$ and for $1 \leq k \leq n$, define

$$S_k(\omega) = \sum_{1 \leq j \leq k} \epsilon_j(\omega) x_j, \qquad \omega \in \Omega.$$

Let

$$A = \left\{\max_{1 \leq k \leq n} \|S_k\| \geq a\right\}, \quad B = \{\|S_n\| \geq a\}, \quad C = \{\|S_n\| \geq 2a\}.$$

Lévy's inequality tells us that $P(A) \leq 2P(B)$.

For $1 \leq k \leq n$, let

$$A_k = \bigcap_{0 \leq j < k} \{\|S_j\| < a\} \cap \{\|S_k\| \geq a\}$$

and

$$C_k = \{\|S_n - S_{k-1}\| \geq a\}.$$

If $\omega \in A_k \cap C$, then

$$\|S_n(\omega) - S_{k-1}(\omega)\| \geq \|S_n(\omega)\| - \|S_{k-1}(\omega)\| \geq 2a - a = a,$$

hence $A_k \cap C \subset C_k$. Then because $C \subset A$ and because A is the disjoint union of $A_1, \ldots, A_n$,

$$P(C) = P(A \cap C) = P\left(\bigcup_{k=1}^n (A_k \cap C)\right) = \sum_{k=1}^n P(A_k \cap C) \leq \sum_{k=1}^n P(A_k \cap C_k).$$

Let $1 \leq k \leq n$. $P(\epsilon_k^2 = 1) = 1$, so for almost all $\omega \in \Omega$,

$$\left\|\sum_{j=k}^n \epsilon_j(\omega) x_j\right\| = \left\|\epsilon_k(\omega) \sum_{j=k}^n \epsilon_j(\omega) x_j\right\| = \left\|x_k + \sum_{j=k+1}^n \epsilon_k(\omega)\epsilon_j(\omega) x_j\right\|.$$

Thus, for

$$D_k = \left\{\left\|x_k + \sum_{j=k+1}^n \epsilon_k \epsilon_j x_j\right\| \geq a\right\},$$

[3] Joe Diestel, Hans Jarchow, and Andrew Tonge, *Absolutely Summing Operators*, p. 214, Lemma 11.4.

we have

$$P(C_k \triangle D_k) = 0.$$

Let $(\delta_1, \dots, \delta_n) \in \{+1, -1\}^n$. On the one hand, because $\delta_j^2 = 1$ and using that $\epsilon_1, \dots, \epsilon_n$ are independent,

$$\begin{aligned}
&P(\epsilon_1 = \delta_1, \dots, \epsilon_k = \delta_k, \epsilon_k\epsilon_{k+1} = \delta_{k+1}, \dots, \epsilon_k\epsilon_n = \delta_n) \\
=&P(\epsilon_1 = \delta_1, \dots, \epsilon_k = \delta_k, \epsilon_{k+1} = \delta_k\delta_{k+1}, \dots, \epsilon_n = \delta_k\delta_n) \\
=&P(\epsilon_1 = \delta_1) \cdots P(\epsilon_k = \delta_k) P(\epsilon_{k+1} = \delta_k\delta_{k+1}) \cdots P(\epsilon_n = \delta_k\delta_n) \\
=&2^{-n}.
\end{aligned}$$

On the other hand, for $k + 1 \leq j \leq n$ we have

$$\begin{aligned}
&P(\epsilon_k\epsilon_j = \delta_j) \\
=&P(\epsilon_k\epsilon_j = \delta_j | \epsilon_k = 1) P(\epsilon_k = 1) + P(\epsilon_k\epsilon_j = \delta_j | \epsilon_k = -1) P(\epsilon_k = -1) \\
=&\frac{1}{2} P(\epsilon_j = \delta_j) + \frac{1}{2} P(\epsilon_j = -\delta_j) \\
=&\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \\
=&\frac{1}{2},
\end{aligned}$$

and hence

$$P(\epsilon_1 = \delta_1) \cdots P(\epsilon_k = \delta_k) P(\epsilon_k\epsilon_{k+1} = \delta_{k+1}) \cdots P(\epsilon_k\epsilon_n = \delta_n) = 2^{-n}.$$

Therefore, for each $1 \leq k \leq n$ and for each $(\delta_1, \dots, \delta_n) \in \{+1, -1\}^n$,

$$\begin{aligned}
&P(\epsilon_1 = \delta_1, \dots, \epsilon_k = \delta_k, \epsilon_k\epsilon_{k+1} = \delta_{k+1}, \dots, \epsilon_k\epsilon_n = \delta_n) \\
=&P(\epsilon_1 = \delta_1) \cdots P(\epsilon_k = \delta_k) P(\epsilon_k\epsilon_{k+1} = \delta_{k+1}) \cdots P(\epsilon_k\epsilon_n = \delta_n).
\end{aligned}$$

But for almost all $\omega \in \Omega$,

$$(\epsilon_1(\omega), \dots, \epsilon_k(\omega), \epsilon_k(\omega)\epsilon_{k+1}(\omega), \dots, \epsilon_k(\omega)\epsilon_n(\omega)) \in \{+1, -1\}^n,$$

so it follows that

$$\epsilon_1, \dots, \epsilon_k, \epsilon_k\epsilon_{k+1}, \dots, \epsilon_k\epsilon_n$$

are independent random variables. We check that $A_k \in \sigma(\epsilon_1, \dots, \epsilon_k)$ and $D_k \in \sigma(\sigma_k\sigma_{k+1}, \dots, \sigma_k\sigma_n)$, and what we have just established means that these σ-algebras are independent, so

$$P(A_k \cap D_k) = P(A_k)P(D_k).$$

But

$$A_k \cap (C_k \triangle D_k) = (A_k \cap C_k) \triangle (A_k \cap D_k),$$

so, because $P(C_k \triangle D_k) = 0$,

$$P(A_k \cap C_k) = P(A_k \cap D_k) = P(A_k)P(D_k) = P(A_k)P(C_k).$$

We had already established that $P(C) \leq \sum_{k=1}^n P(A_k \cap C_k)$. Using this with the above, and the fact that A is the disjoint union of $A_1, \ldots, A_n$, we obtain

$$\begin{aligned}
P(C) &\leq \sum_{k=1}^n P(A_k \cap C_k) \\
&= \sum_{k=1}^n P(A_k)P(C_k) \\
&\leq \left(\sum_{k=1}^n P(A_k)\right) \max_{1\leq k\leq n} P(C_k) \\
&= P\left(\bigcup_{k=1}^n A_k\right) \max_{1\leq k\leq n} P(C_k) \\
&= P(A) \max_{1\leq k\leq n} P(C_k).
\end{aligned}$$

As we stated before, we have from Lévy's inequality that $P(A) \leq 2P(B)$, with which

$$P(C) \leq 2P(B) \max_{1\leq k\leq n} P(C_k).$$

To prove the claim it thus suffices to show that

$$\max_{1\leq k\leq n} P(C_k) \leq 2P(B).$$

Let $1 \leq k \leq n$. For $\delta = (\delta_1, \ldots, \delta_{k-1}) \in \{+1, -1\}^{k-1}$, let let $H_{k,\delta,+}$ be those $(t_1, \ldots, t_n) \in \mathbb{R}^n$ such that (i) for each $1 \leq j \leq k-1$, $t_j = \delta_j$, (ii) $\left\|\sum_{j=k}^n t_j x_j\right\| \geq a$, and (iii)

$$\left\|\sum_{j=1}^n t_j x_j\right\| \geq a,$$

and let $H_{k,\delta,-}$ be those $(t_1, \ldots, t_n) \in \mathbb{R}^n$ satisfying (i) and (ii) and

$$\left\|\sum_{j=1}^{k-1} t_j x_j - \sum_{j=k}^n t_j x_j\right\| \geq a.$$

Let

$$X = (\epsilon_1, \ldots, \epsilon_n) : \Omega \to \mathbb{R}^n$$

and let

$$X_k = (\epsilon_1, \ldots, \epsilon_{k-1}, -\epsilon_k, \ldots, -\epsilon_n) : \Omega \to \mathbb{R}^n,$$

which have the same distribution because $\epsilon_1, \ldots, \epsilon_n$ are independent and symmetric. Then

$$\begin{aligned} P(X^{-1}(H_{k,\delta,+})) &= (X_* P)(H_{k,\delta,+}) \\ &= (X_{k*} P)(H_{k,\delta,+}) \\ &= P(X_k^{-1}(H_{k,\delta,+})) \\ &= P(X^{-1}(H_{k,\delta,-})). \end{aligned}$$

Set

$$C_{k,\delta,+} = \{X \in H_{k,\delta,+}\}, \qquad C_{k,\delta,-} = \{X \in H_{k,\delta,-}\},$$

for which we thus have

$$P(C_{k,\delta,+}) = P(C_{k,\delta,-}).$$

We can write $C_{k,\delta,+}$ and $C_{k,\delta,-}$ as

$$C_{k,\delta,+} = \left(\bigcap_{0 \le j < k} \{\epsilon_j = \delta_j\} \right) \cap C_k \cap \{\|S_n\| \ge a\}$$

and

$$C_{k,\delta,-} = \left(\bigcap_{0 \le j < k} \{\epsilon_j = \delta_j\} \right) \cap C_k \cap \{\|S_n - 2S_{k-1}\| \ge a\}.$$

If $\omega \in C_k$ then, because $\|S_n(\omega) - S_{k-1}(\omega)\| \ge a$,

$$\begin{aligned} 2a &\le 2\,\|S_n(\omega) - S_{k-1}(\omega)\| \\ &= \|S_n(\omega) + (S_n(\omega) - 2S_{k-1}(\omega))\| \\ &\le \|S_n(\omega)\| + \|S_n(\omega) - 2S_{k-1}(\omega)\|, \end{aligned}$$

so at least one of the inequalities $\|S_n(\omega)\| \ge a$ and $\|S_n(\omega) - 2S_{k-1}(\omega)\| \ge a$ is true, and hence

$$C_k \subset \{\|S_n\| \ge a\} \cup \{\|S_n - 2S_{k-1}\| \ge a\}.$$

It follows that

$$C_k \cap \left(\bigcap_{0 \le j < k} \{\epsilon_j = \delta_j\} \right) = C_{k,\delta,+} \cup C_{k,\delta,-}.$$

Therefore, using the fact that for almost all $\omega \in \Omega$,

$$(\epsilon_1(\omega), \ldots, \epsilon_{k-1}(\omega)) \in \{+1, -1\}^{k-1},$$

and

$$C_{k,\delta,+} = \left(\bigcap_{0 \le j < k} \{\epsilon_j = \delta_j\} \right) \cap C_k \cap B,$$

we get

$$
\begin{aligned}
P(C_k) &= \sum_\delta P\left(C_k \cap \bigcap_{0\le j<k} \{\epsilon_j = \delta_j\}\right)\\
&= \sum_\delta P(C_{k,\delta,+} \cup C_{k,\delta,-})\\
&= 2\sum_\delta P(C_{k,\delta,+})\\
&\le 2\sum_\delta P\left(B \cap \bigcap_{0\le j<k} \{\epsilon_j = \delta_j\}\right)\\
&= 2P(B),
\end{aligned}
$$

and thus

$$\max_{1\le k\le n} P(C_k) \le 2P(B),$$

which proves the claim. □

3 Kahane's inequality

By $E(X)^r$ we mean $(E(X))^r$. The following is **Kahane's inequality**.[4]

Theorem 3 (Kahane's inequality)**.** *For $0 < p, q < \infty$, there is some $K_{p,q} > 0$ such that if X is a real or complex Banach space and $x_k \in X$, $k \ge 1$, then for each n,*

$$E\left(\left\|\sum_{k=1}^n \epsilon_k x_k\right\|^q\right)^{1/q} \le K_{p,q} \cdot E\left(\left\|\sum_{k=1}^n \epsilon_k x_k\right\|^p\right)^{1/p}.$$

Proof. Suppose that $0 < p < q < \infty$; when $p \ge q$ the claim is immediate with $K_{p,q} = 1$. Let

$$M = E\left(\left\|\sum_{k=1}^n \epsilon_k x_k\right\|^p\right)^{1/p};$$

if $M = 0$ we check that the claim is $0 \le K_{p,q} \cdot 0$, which is true for, say, $K_{p,q} = 1$. Otherwise, $M > 0$, and let $u_k = \frac{x_k}{M}$, $1 \le k \le n$, for which

$$E\left(\left\|\sum_{k=1}^n \epsilon_k u_k\right\|^p\right) = E\left(\left\|\sum_{k=1}^n \epsilon_k \frac{x_k}{M}\right\|^p\right) = 1. \tag{1}$$

Using Chebyshev's inequality,

$$P\left(\left\|\sum_{k=1}^n \epsilon_k u_k\right\| \ge 8^{1/p}\right) = P\left(\left\|\sum_{k=1}^n \epsilon_k u_k\right\|^p \ge 8\right) \le \frac{1}{8}E\left(\left\|\sum_{k=1}^n \epsilon_k u_k\right\|^p\right) = \frac{1}{8}.$$

[4] Joe Diestel, Hans Jarchow, and Andrew Tonge, *Absolutely Summing Operators*, p. 211, Theorem 11.1.

Assume for induction that for some $l \geq 0$ we have

$$P\left(\left\|\sum_{k=1}^{n} \epsilon_k u_k\right\| \geq 2^l \cdot 8^{1/p}\right) \leq \frac{1}{4} \cdot 2^{-2^l}; \tag{2}$$

the above shows that this is true for $l = 0$. Applying Theorem 2 and then (2),

$$P\left(\left\|\sum_{k=1}^{n} \epsilon_k u_k\right\| \geq 2^{l+1} \cdot 8^{1/p}\right) \leq 4\left(P\left(\left\|\sum_{k=1}^{n} \epsilon_k u_k\right\| \geq 2^l \cdot 8^{1/p}\right)\right)^2 \leq \frac{1}{4} \cdot 2^{-2^{l+1}},$$

which shows that (2) is true for all $l \geq 0$.

Generally, for $0 < q < \infty$, if $X : \Omega \to \mathbb{R}$ is a random variable for which $P(X \geq 0) = 1$, then

$$E(X^q) = \int_0^\infty q s^{q-1} P(X \geq s) ds;$$

the right-hand side is finite if and only if $X \in L^q(P)$. Using this,

$$E\left(\left\|\sum_{k=1}^{n} \epsilon_k u_k\right\|^q\right) = \int_0^\infty q s^{q-1} P\left(\left\|\sum_{k=1}^{n} \epsilon_k u_k\right\| \geq s\right) ds. \tag{3}$$

Let $\alpha_0 =$ and for $l \geq 1$ let $\alpha_l = 2^{l-1} \cdot 8^{1/p}$, and define

$$f(s) = q s^{q-1} P\left(\left\|\sum_{k=1}^{n} \epsilon_k u_k\right\| \geq s\right), \qquad s \geq 0.$$

Using (3) and then (2),

$$\begin{aligned}
E\left(\left\|\sum_{k=1}^{n} \epsilon_k u_k\right\|^q\right) &= \int_0^\infty f(s) ds \\
&= \int_0^{\alpha_1} f(s) ds + \sum_{l=0}^{\infty} \int_{\alpha_{l+1}}^{\alpha_{l+2}} f(s) ds \\
&\leq \int_0^{\alpha_1} q s^{q-1} ds + \sum_{l=0}^{\infty} \int_{\alpha_{l+1}}^{\alpha_{l+2}} q s^{q-1} P\left(\left\|\sum_{k=1}^{n} \epsilon_k u_k\right\| \geq \alpha_{l+1}\right) ds \\
&\leq \alpha_1^q + \sum_{l=0}^{\infty} \int_{\alpha_{l+1}}^{\alpha_{l+2}} q s^{q-1} \frac{1}{4} \cdot 2^{-2^l} ds \\
&= 8^{q/p} + \frac{1}{4} \sum_{l=0}^{\infty} 2^{-2^l} (\alpha_{l+2}^q - \alpha_{l+1}^q),
\end{aligned}$$

and we define $K_{p,q}$ by taking $K_{p,q}^q$ to be equal to the above. Thus

$$E\left(\left\|\sum_{k=1}^{n} \epsilon_k u_k\right\|^q\right)^{1/q} \leq K_{p,q},$$

and therefore, by (1),

$$E\left(\left\|\sum_{k=1}^{n}\epsilon_k u_k\right\|^q\right)^{1/q} \leq K_{p,q}\cdot E\left(\left\|\sum_{k=1}^{n}\epsilon_k u_k\right\|^p\right)^{1/p}.$$

Finally, as $u_k = \frac{x_k}{M}$,

$$E\left(\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^q\right)^{1/q} \leq K_{p,q}\cdot E\left(\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^p\right)^{1/p},$$

which proves the claim. □

In the above proof of Kahane's inequality, for $p = 1$ and $q = 2$ we have

$$\begin{aligned} K_{1,2}^2 &= 8^2 + \frac{1}{4}\sum_{l=0}^{\infty} 2^{-2^l}(\alpha_{l+2}^2 - \alpha_{l+1}^2) \\ &= 64 + 16\sum_{l=0}^{\infty} 2^{-2^l}(2^{2l+2} - 2^{2l}) \\ &= 64 + 48\sum_{l=0}^{\infty} 2^{-2^l}2^{2l}, \end{aligned}$$

for which

$$K_{1,2} = 14.006\ldots.$$

In fact, the inequality is true with $K_{1,2} = \sqrt{2} = 1.414\ldots$.[5]

[5] R. Latała and K. Oleszkiewicz, *On the best constant in the Khinchin-Kahane inequality*, Studia Math. **109** (1994), no. 1, 101–104.