Topological spaces and neighborhood filters

Jordan Bell
jordan.bell@gmail.com
Department of Mathematics, University of Toronto

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If X is a set, a *filter* on X is a set $\mathcal{F}$ of subsets of X such that $\emptyset \notin \mathcal{F}$; if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$; if $A \subseteq X$ and there is some $B \in \mathcal{F}$ such that $B \subseteq A$, then $A \in \mathcal{F}$. For example, if $x \in X$ then the set of all subsets of X that include x is a filter on X.[1] A *basis* for the filter $\mathcal{F}$ is a subset $\mathcal{B} \subseteq \mathcal{F}$ such that if $A \in \mathcal{F}$ then there is some $B \in \mathcal{B}$ such that $B \subseteq A$.

If X is a set, a *topology* on X is a set $\mathcal{O}$ of subsets of X such that: $\emptyset, X \in \mathcal{O}$; if $U_\alpha \in \mathcal{O}$ for all $\alpha \in I$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{O}$; if I is finite and $U_\alpha \in \mathcal{O}$ for all $\alpha \in I$, then $\bigcap_{\alpha \in I} U_\alpha \in \mathcal{O}$. If $N \subseteq X$ and $x \in X$, we say that N is a *neighborhood* of x if there is some $U \in \mathcal{O}$ such that $x \in U \subseteq N$. In particular, an open set is a neighborhood of every element of itself. A *basis* for a topology $\mathcal{O}$ is a subset $\mathcal{B}$ of $\mathcal{O}$ such that if $x \in X$ then there is some $B \in \mathcal{B}$ such that $x \in B$, and such that if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.[2]

On the one hand, suppose that X is a topological space with topology $\mathcal{O}$. For each $x \in X$, let $\mathcal{F}_x$ be the set of neighborhoods of x; we call $\mathcal{F}_x$ the *neighborhood filter* of x. It is straightforward to verify that $\mathcal{F}_x$ is a filter for each $x \in X$. If $N \in \mathcal{F}_x$, there is some $U \in \mathcal{F}_x$ that is open, and for each $y \in U$ we have $N \in \mathcal{F}_y$.

On the other hand, suppose X is a set, for each $x \in X$ there is some filter $\mathcal{F}_x$, and: if $N \in \mathcal{F}_x$ then $x \in N$; if $N \in \mathcal{F}_x$ then there is some $U \in \mathcal{F}_x$ such that if $y \in U$ then $N \in \mathcal{F}_y$. We define $\mathcal{O}$ in the following way: The elements U of $\mathcal{O}$ are those subsets of X such that if $x \in U$ then $U \in \mathcal{F}_x$. Vacuously, $\emptyset \in \mathcal{O}$, and it is immediate that $X \in \mathcal{O}$. If $U_\alpha \in \mathcal{O}$, $\alpha \in I$ and $x \in U = \bigcup_{\alpha \in I} U_\alpha$ then there is at least one $\alpha \in I$ such that $x \in U_\alpha$ and so $U_\alpha \in \mathcal{F}_x$. As $x \in U_\alpha \subseteq U$ and $\mathcal{F}_x$ is a filter, we get $U \in \mathcal{F}_x$. If I is finite and $U_\alpha \in I$, $\alpha \in I$, let $U = \bigcap_{\alpha \in I} U_\alpha$. If $x \in U$, then for each $\alpha \in I$, $x \in U_\alpha$, and hence for each $\alpha \in I$, $U_\alpha \in \mathcal{F}_x$. As $\mathcal{F}_x$ is a filter, the intersection of any two elements of it is an element of it, and thus the intersection of finitely many elements of it is an element of it, so $U \in \mathcal{F}_x$, showing that $U \in \mathcal{O}$. This shows that $\mathcal{O}$ is a topology. We will show that a set N is a neighborhood of a point x if and only if $N \in \mathcal{F}_x$.

If $N \in \mathcal{F}_x$, then let $V = \{y \in N : N \in \mathcal{F}_y\}$. There is some $U_0 \in \mathcal{F}_x$ such

[1] cf. François Trèves, *Topological Vector Spaces, Distributions and Kernels*, p. 6.
[2] cf. James R. Munkres, *Topology*, second ed., p. 78.

that if $y \in U_0$ then $N \in \mathcal{F}_y$. If $y \in U_0$ then $N \in \mathcal{F}_y$, which implies that $y \in N$, and hence $U_0 \subseteq V$. $U_0 \subseteq V$ and $U_0 \in \mathcal{F}_x$ imply that $V \in \mathcal{F}_x$, which implies that $x \in V$. If $y \in V$ then $N \in \mathcal{F}_y$, and hence there is some $U \in \mathcal{F}_y$ such that if $z \in U$ then $N \in \mathcal{F}_z$. If $z \in U$ then $N \in \mathcal{F}_z$, which implies that $z \in N$, and hence $U \subseteq V$. $U \subseteq V$ and $U \in \mathcal{F}_y$ imply that $V \in \mathcal{F}_y$. Thus, if $y \in V$ then $V \in \mathcal{F}_y$, which means that V is open, $x \in V \subseteq N$ tells us that N is a neighborhood of x.

If a set N is a neighborhood of a point x, then there is some open set U with $x \in U \subseteq N$. U being open means that if $y \in U$ then $U \in \mathcal{F}_y$. As $x \in U$ we get $U \in \mathcal{F}_x$, and as $U \subset N$ we get $N \in \mathcal{F}_x$. Therefore a set N is a neighborhood of a point x if and only if $N \in \mathcal{F}_x$.

In conclusion: If X is a topological space and for each $x \in X$ we define $\mathcal{F}_x$ to be the neighborhood filter of x, then these filters satisfy the two conditions that if $N \in \mathcal{F}_x$ then $x \in N$ and that if $N \in \mathcal{F}_x$ there is some $U \in \mathcal{F}_x$ such that if $y \in U$ then $N \in \mathcal{F}_y$. In the other direction, if X is a set and for each point $x \in X$ there is a filter $\mathcal{F}_x$ and the filters satisfy these two conditions, then there is a topology on X such that these filters are precisely the neighborhood filters of each point.