p-adic test functions

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1 $\mathbb{Q}_p^n$

Let p be prime, let $N_p = \{0, \ldots, p-1\}$, and let $\mathbb{Q}_p \subset \prod_{\mathbb{Z}} N_p$ be the p-adic numbers. For $x \in \mathbb{Q}_p$ let

$$v_p(x) = \inf\{k \in \mathbb{Z} : x(k) \neq 0\}, \qquad |x|_p = p^{-v_p(x)}.$$

For $r > 0$ and $a \in \mathbb{Q}_p$ let

$$B_{\leq r}(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\},$$

and let

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : v_p(x) \geq 0\} = B_{\leq 1}(0).$$

For $l \in \mathbb{Z}$, $v_p(p^l) = l$, $|p^l|_p = p^{-l}$, and

$$p^l \mathbb{Z}_p = \{x \in \mathbb{Q}_p : v_p(x) \geq l\} = B_{\leq p^{-l}}(0).$$

Let μ be the Haar measure on the additive group $\mathbb{Q}_p$ with $\mu(\mathbb{Z}_p) = 1$. It is a fact that if A is a Borel set in $\mathbb{Q}_p$ and $x \in \mathbb{Q}_p$ then

$$\mu(x \cdot A) = |x|_p \mu(A).$$

In particular, for $l \in \mathbb{Z}$ and $x = p^l$, $\mu(p^l \cdot A) = |p^l|_p \mu(A) = p^{-l}\mu(A)$ and so

$$\mu(p^l \mathbb{Z}_p) = p^{-l}, \qquad l \in \mathbb{Z}.$$

Let $n \geq 1$. For $x \in \mathbb{Q}_p^n$ let

$$|x|_p = \max\{|x_j|_p : 1 \leq j \leq n\}.$$

For $r > 0$ and $a \in \mathbb{Q}_p$ let

$$B^n_{\leq r}(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq r\} = \prod_{j=1}^n B_{\leq r}(a_j).$$

For $l \in \mathbb{Z}$,

$$p^l \mathbb{Z}_p^n = p^l B_{\leq 1}^n(0) = B_{\leq p^{-l}}^n(0) = (p^l \mathbb{Z}_p)^n.$$

Let $\mu_n = \bigotimes_{j=1}^n \mu$, the product measure on the Borel σ-algebra of $\mathbb{Q}_p^n$. Then

$$\mu_n(p^l \mathbb{Z}_p^n) = \prod_{j=1}^n \mu(p^l \mathbb{Z}_p) = \prod_{j=1}^n p^{-l} = p^{-nl}.$$

2 Locally constant functions

Let O be an open set in $\mathbb{Q}_p^n$. A function $\psi : O \to \mathbb{C}$ is called **locally constant** if for each $x \in O$ there is some neighborhood N_x of x such that $\psi(y) = \psi(x)$ for $y \in N_x$. In this case, there is some $l(x) \in \mathbb{Z}$ such that $x + p^{l(x)} \mathbb{Z}_p^n \subset N_x$, and so

$$\psi(x+h) = \psi(x), \qquad x \in O, \quad h \in p^{l(x)} \mathbb{Z}_p^n.$$

It is immediate that a locally constant function is continuous. Let $\mathcal{E}(O)$ be the collection of locally constant functions $O \to \mathbb{C}$.

Because locally constant functions are used often in p-adic analysis, it is worthwhile working out some facts about them.[1]

Lemma 1. *If $\psi \in \mathcal{E}(\mathbb{Q}_p^n)$ and K is a compact set in $\mathbb{Q}_p^n$, then there is some $l \in \mathbb{Z}$ such that*

$$\psi(x+h) = \psi(x), \qquad x \in K, \qquad h \in p^l \mathbb{Z}_p^n.$$

Proof. Because K is compact it is bounded and so is contained in $p^N \mathbb{Z}_p^n$ for some $m \in \mathbb{Z}$. Now, $\{x + p^{l(x)} \mathbb{Z}_p^n : x \in p^N \mathbb{Z}_p^n\}$ is an open cover of $p^N \mathbb{Z}_p^n$, and because $p^N \mathbb{Z}_p^n$ is compact there are $x^1, \ldots, x^m \in p^N \mathbb{Z}_p^n$ such that $K \subset \bigcup_{k=1}^m (x^k + p^{l(x^k)} \mathbb{Z}_p^n)$. We further specify that these sets are pairwise disjoint, which we can because two balls in $\mathbb{Q}_p^n$ have nonempty intersection if and only if one is contained in another.[2] Let $l = \max\{l(x^k) : 1 \leq k \leq m\}$. For $x \in K$ there is some k for which $x \in x^k + p^{l(x^k)} \mathbb{Z}_p^n$, and as $x - x^k \in p^{l(x^k)} \mathbb{Z}_p^n$, for $h \in p^l \mathbb{Z}_p^n$,

$$|x - x^k + h|_p \leq \max(|x - x^k|_p, |h|_p) \leq \max(p^{-l(x^k)}, p^{-l}) = p^{-l(x^k)},$$

i.e. $x - x^k + h \in p^{l(x^k)} \mathbb{Z}_p^n$. Then using that ψ is locally constant, with $O = x^k + p^{l(x^k)} \mathbb{Z}_p^n$,

$$\psi(x+h) = \psi(x^k + (x - x^k + h)) = \psi(x^k).$$

And $x - x^k \in p^{l(x^k)} \mathbb{Z}_p^n$ means that $\psi(x^k + x - x^k) = \psi(x^k)$, i.e. $\psi(x) = \psi(x^k)$, showing $\psi(x+h) = \psi(x)$. □

[1] S. Albeverio, A. Yu Khrennikov, and V. M. Shelkovich, *Theory of p-adic Distributions: Linear and Nonlinear Models*, p. 55, Lemma 4.2.1.

[2] This is not transparent but is straightforward to check.

3 p-adic test functions

Let $\mathcal{D}(\mathbb{Q}_p^n)$ be the set of those $\psi \in \mathcal{E}(\mathbb{Q}_p^n)$ such that $\operatorname{supp}\psi$ is a compact set. Elements of $\mathcal{D}(\mathbb{Q}_p^n)$ are called **p-adic test functions**.