The Poisson summation formula, the sampling theorem, and Dirac combs

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1 Poisson summation formula

Let $\mathcal{S}(\mathbb{R})$ be the set of all infinitely differentiable functions f on $\mathbb{R}$ such that for all nonnegative integers m and n,

$$\nu_{m,n}(f) = \sup_{x\in\mathbb{R}} |x|^m |f^{(n)}(x)|$$

is finite. For each m and n, $\nu_{m,n}$ is a seminorm. This is a countable collection of seminorms, and $\mathcal{S}(\mathbb{R})$ is a Fréchet space. (One has to prove for $v_k \in \mathcal{S}(\mathbb{R})$ that if for each fixed m, n the sequence $\nu_{m,n}(f_k)$ is a Cauchy sequence (of numbers), then there exists some $v \in \mathcal{S}(\mathbb{R})$ such that for each m, n we have $\nu_{m,n}(f - f_k) \to 0$.) I mention that $\mathcal{S}(\mathbb{R})$ is a Fréchet space merely because it is satisfying to give a structure to a set with which we are working: if I call this set Schwartz space I would like to know what type of space it is, in the same sense that an L^p space is a Banach space.

For $f \in \mathcal{S}(\mathbb{R})$, define

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dx,$$

and then

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) d\xi.$$

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For $\phi \in C^\infty(\mathbb{T})$, define

$$\hat{\phi}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} \phi(t) dt,$$

and then

$$\phi(t) = \sum_{n\in\mathbb{Z}} \hat{\phi}(n) e^{int}.$$

For $f \in \mathcal{S}(\mathbb{R})$ and $\lambda \neq 0$, define $\phi_\lambda : \mathbb{T} \to \mathbb{C}$ by

$$\phi_\lambda(t) = 2\pi \sum_{j \in \mathbb{Z}} f_\lambda(t + 2\pi j),$$

where $f_\lambda(x) = \lambda f(\lambda x)$. We have $\phi_\lambda \in C^\infty(\mathbb{T})$.

For $n \in \mathbb{Z}$,

$$\begin{aligned}
\widehat{\phi_\lambda}(n) &= \frac{1}{2\pi} \int_0^{2\pi} e^{-int} \cdot 2\pi \sum_{j \in \mathbb{Z}} f_\lambda(t + 2\pi j) dt \\
&= \sum_{j \in \mathbb{Z}} \int_0^{2\pi} e^{-int} f_\lambda(t + 2\pi j) dt \\
&= \int_{\mathbb{R}} e^{-inx} f_\lambda(x) dx \\
&= \widehat{f_\lambda}(n).
\end{aligned}$$

One checks that $\widehat{f_\lambda}(\xi) = \hat{f}\left(\frac{\xi}{\lambda}\right)$, so $\widehat{\phi_\lambda}(n) = \hat{f}\left(\frac{n}{\lambda}\right)$. Thus for $t \in \mathbb{T}$,

$$\phi_\lambda(t) = \sum_{n \in \mathbb{Z}} \hat{f}\left(\frac{n}{\lambda}\right) e^{int}.$$

Therefore, if $f \in \mathcal{S}(\mathbb{R})$ then for each $t \in \mathbb{T}$,

$$2\pi \sum_{j \in \mathbb{Z}} f_\lambda(t + 2\pi j) = \sum_{n \in \mathbb{Z}} \hat{f}\left(\frac{n}{\lambda}\right) e^{int},$$

where $f_\lambda(x) = \lambda f(\lambda x)$. This is the *Poisson summation formula*.

2 Nyquist-Shannon sampling theorem

Let $f \in \mathcal{S}(\mathbb{R})$. Suppose that there is some L such that $f(x) = 0$ if $|x| > L$ (namely, f is *spacelimited*). We can in fact choose λ so that all terms with $j \neq 0$ on the left-hand side of the Poisson summation formula are zero, which then expresses $f_\lambda(t)$ in terms of discrete samples of the Fourier transform of f.

If there is an L such that $\hat{f}(\xi) = 0$ for $|\xi| > L$, we say that f is *bandlimited*. (For $f : \mathbb{T} \to \mathbb{C}$ this correspond to f being a trigonometric polynomial.) We can prove a dual Poisson formula and apply it to bandlimited functions, but instead I am going to give a direct argument following Charles L. Epstein, *Introduction to the Mathematics of Medical Imaging*, second ed., which is a good reference for sampling.

Suppose that $f \in \mathcal{S}(\mathbb{R})$ and $f(\xi) = 0$ for $|\xi| > L$.[1] For $-\pi \leq \xi \leq \pi$, define

[1] In fact, if $f, \hat{f} \in L^1(\mathbb{R})$ and f is bandlimited then it follows that f is infinitely differentiable, although it does not follow that f is Schwartz.

$F : \mathbb{T} \to \mathbb{C}$ by[2]

$$F(\xi) = \hat{f}\left(\frac{\xi L}{\pi}\right).$$

For $n \in \mathbb{Z}$,

$$\begin{aligned}
\widehat{F}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\xi} F(\xi) d\xi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\xi} \hat{f}\left(\frac{\xi L}{\pi}\right) d\xi \\
&= \frac{1}{2L}\int_{-L}^{L} e^{-in\frac{\pi t}{L}} \hat{f}(t) dt.
\end{aligned}$$

On the other hand, for $x \in \mathbb{R}$, using the fact that f is bandlimited,

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{f}(\xi)d\xi = \frac{1}{2\pi}\int_{-L}^{L} e^{ix\xi}\hat{f}(\xi)d\xi.$$

Therefore for $n \in \mathbb{Z}$,

$$\widehat{F}(n) = \frac{\pi}{L} f\left(\frac{-n\pi}{L}\right).$$

As $F \in C^\infty(\mathbb{T})$, for $t \in \mathbb{T}$ we have

$$F(t) = \sum_{n\in\mathbb{Z}} \widehat{F}(n) e^{int} = \sum_{n\in\mathbb{Z}} \frac{\pi}{L} f\left(\frac{-n\pi}{L}\right) e^{int}.$$

Hence for $-L \leq \xi \leq L$,

$$\hat{f}(\xi) = F\left(\frac{\pi\xi}{L}\right) = \sum_{n\in\mathbb{Z}} \frac{\pi}{L} f\left(\frac{-n\pi}{L}\right) e^{\frac{in\pi\xi}{L}}.$$

For $0 \leq \eta < L$, let $\phi : \mathbb{R} \to \mathbb{R}$ satisfy

$$\hat{\phi}(\xi) = \begin{cases} 1 & |\xi| \leq L - \eta \\ 0 & |\xi| > L. \end{cases}$$

For instance, for $\eta = 0$ let $\hat{\phi}$ be the characteristic function on $[-L, L]$. But if $\eta > 0$ we can choose $\hat{\eta}$ to be smooth rather than a characteristic function. If $\hat{f}(\xi) = 0$ for $|\xi| > L - \eta$ (rather than just for $|\xi| > L$), then $\hat{f} = \hat{f}\hat{\phi}$, because ϕ

[2] We are defining a periodic function F from a function $\hat{f}$ that is not periodic, but rather which becomes 0 after L.

is equal to 1 on the support of $\hat{f}$. Then for $x \in \mathbb{R}$,

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(xi) \hat{\phi}(\xi) d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \sum_{n\in\mathbb{Z}} \frac{\pi}{L} f\left(\frac{-n\pi}{L}\right) e^{\frac{in\pi\xi}{L}} \hat{\phi}(\xi) d\xi \\ &= \frac{1}{2L} \sum_{n\in\mathbb{Z}} f\left(\frac{-n\pi}{L}\right) \int_{\mathbb{R}} e^{ix\xi + \frac{in\pi\xi}{L}} \hat{\phi}(\xi) d\xi \\ &= \frac{1}{2L} \sum_{n\in\mathbb{Z}} f\left(\frac{-n\pi}{L}\right) \cdot 2\pi \cdot \phi\left(x + \frac{n\pi}{L}\right) \\ &= \frac{\pi}{L} \sum_{n\in\mathbb{Z}} f\left(\frac{-n\pi}{L}\right) \phi\left(x + \frac{n\pi}{L}\right). \end{aligned}$$

$\hat{\phi}$ can be chosen so that $\phi\left(x + \frac{n\pi}{L}\right)$ quickly goes to 0 as $n \to \infty$. This means that fewer terms of the above series need to be calculated to get a good approximation of $f(x)$. For $\hat{\phi}$ the characteristic function on $[-L, L]$, we have

$$\phi(x) = \frac{1}{2\pi} \int_{-L}^{L} e^{ix\xi} d\xi = \frac{\sin(xL)}{\pi x} = \frac{L}{\pi} \text{sinc}(xL),$$

where $\text{sinc}(x) = \frac{\sin x}{x}$. Then we obtain, for $x \in \mathbb{R}$,

$$f(x) = \sum_{n\in\mathbb{Z}} f\left(\frac{-n\pi}{L}\right) \text{sinc}\left(x + \frac{n\pi}{L}\right),$$

or

$$f(x) = \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{L}\right) \text{sinc}\left(x - \frac{n\pi}{L}\right).$$

Thus if f is bandlimited, we can sample f at a sequence of points and using these values reconstruct $f(x)$ for any $x \in \mathbb{R}$. This fact is the *Nyquist-Shannon sampling theorem*, and the actual formula above is the *Whittaker-Shannon interpolation formula*.

A pleasant exposition of sampling is given in *Sampling: What Nyquist DidnÕt Say, and What to Do About It*, by Tim Wescott, 2010.

3 Dirac combs, pulse trains

Define a tempered distribution Ш on $\mathbb{R}$ by

$$\text{Ш}(t) = \sum_{n\in\mathbb{Z}} \delta(t - n).$$

This is called a *Dirac comb* or *pulse train*. If $f \in \mathcal{S}(\mathbb{R})$,

$$\text{Ш} f = \sum_{n\in\mathbb{Z}} f(n).$$

The convolution of f and Ш is the function

$$(Ш * f)(t) = \sum_{n\in\mathbb{Z}} f(t-n).$$

Define

$$Ш_T(t) = \frac{1}{T} Ш \left(\frac{t}{T}\right).$$

Because $\delta(t/a) = |a|\delta(t)$ (the *scaling property* of the Dirac delta distribution; we have an absolute value sign because if a is negative then doing a change of variables in the definition of $\delta(t/a)$ we get two negative signs), if $T > 0$ then

$$Ш_T(t) = \sum_{n\in\mathbb{Z}} \frac{1}{T} \cdot \delta\left(\frac{t}{T} - n\right) = \sum_{n\in\mathbb{Z}} \delta(t - nT),$$

and we have

$$Ш_T f = \sum_{n\in\mathbb{Z}} f(nT)$$

and

$$(Ш_T * f)(t) = \sum_{n\in\mathbb{Z}} f(t - nT).$$

$Ш_T(t+T) = Ш_T(t)$, so $Ш_T$ is a periodic distribution on $\mathbb{R}$. It follows that $Ш_T$ is a distribution on $\mathbb{T}$. Folland talks about periodic distributions around p. 298 of his *Real Analysis*, second ed.

Let $\gamma \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \gamma(x)dx = \frac{1}{2\pi}$, and let $\omega = \gamma * \chi_{[0,2\pi)}$. If we have a 2π-periodic distribution F on $\mathbb{R}$ and $\psi \in C^\infty(\mathbb{T})$, we define $F\psi$ to be $F(\omega\psi)$ (ψ is viewed in this instance as a 2π-periodic function on $\mathbb{R}$). For $\psi \in C^\infty(\mathbb{T})$ we have, viewing ψ as a 2π-periodic function on $\mathbb{R}$ (thus $\psi(2n\pi) = \psi(0)$),

$$\begin{aligned}
Ш_{2\pi}(\omega\psi) &= \sum_{n\in\mathbb{Z}} \delta(t - 2n\pi)(\omega\psi) \\
&= \sum_{n\in\mathbb{Z}} \omega(2n\pi)\psi(2n\pi) \\
&= \psi(0) \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \gamma(x)\chi_{[0,2\pi)}(2n\pi - x)dx \\
&= \psi(0) \int_{\mathbb{R}} \gamma(x) \sum_{n\in\mathbb{Z}} \chi_{[0,2\pi)}(2n\pi - x)dx \\
&= \psi(0) \int_{\mathbb{R}} \gamma(x) \\
&= \frac{\psi(0)}{2\pi}.
\end{aligned}$$

Thus as a distribution on $\mathbb{T}$, for $\psi \in C^\infty(\mathbb{T})$ we have $Ш_{2\pi}\psi = \frac{\psi(0)}{2\pi}$. For $n \in \mathbb{Z}$, $\widehat{Ш_{2\pi}}(n) = Ш_{2\pi}e^{-int} = \frac{1}{2\pi}$. Thus the Fourier series of $Ш_{2\pi}$ is

$$\frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{int}.$$