Infinite product measures

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1 Introduction

The usual proof that the product of a collection of probability measures exists uses Fubini's theorem. This is unsatisfying because one ought not need to use Fubini's theorem to prove things having only to do with σ-algebras and measures. In this note I work through the proof given by Saeki of the existence of the product of a collection of probability measures.[1] We speak only about the Lebesgue integral of characteristic functions.

2 Rings of sets and Hopf's extension theorem

If X is a set and $\mathscr{R}$ is a collection of subsets of X, we call $\mathscr{R}$ a **ring of sets** when (i) $\emptyset \in \mathscr{R}$ and (ii) if A and B belong to $\mathscr{R}$ then $A \cup B$ and $A \setminus B$ belong to $\mathscr{R}$. If $\mathscr{R}$ is a ring of sets and $A, B \in \mathscr{R}$, then $A \cap B = A \setminus (A \setminus B) \in \mathscr{R}$. Equivalently, one checks that a collection of subsets $\mathscr{R}$ of X is a ring of sets if and only if (i) $\emptyset \in \mathscr{R}$ and (ii) if A and B belong to $\mathscr{R}$ then $A \triangle B$ and $A \cap B$ belong to $\mathscr{R}$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the **symmetric difference**. One checks that indeed a ring of sets is a ring with addition $\triangle$ and multiplication $\cap$. If $\mathscr{S}$ is a nonempty collection of subsets of X, one proves that there is a unique ring of sets $\mathscr{R}(\mathscr{S})$ that (i) contains $\mathscr{S}$ and (ii) is contained in any ring of sets that contains $\mathscr{S}$. We call $\mathscr{R}(\mathscr{S})$ the **ring of sets generated by** $\mathscr{S}$.

If $\mathscr{A}$ is a ring of subsets of a set X, we call $\mathscr{A}$ an **algebra of sets** when $X \in \mathscr{A}$. Namely, an algebra of sets is a unital ring of sets. If $\mathscr{S}$ is a nonempty collection of subsets of X, one proves that there is a unique algebra of sets $\mathscr{A}(\mathscr{S})$ that (i) contains $\mathscr{S}$ and (ii) is contained in any algebra of sets that contains $\mathscr{S}$. We call $\mathscr{A}(\mathscr{S})$ the **algebra of sets generated by** $\mathscr{S}$.

For a nonempty collection $\mathscr{G}$ of subsets of a set X, we denote by $\sigma(\mathscr{G})$ the smallest σ-algebra of subsets of X such that $\mathscr{G} \subset \sigma(\mathscr{G})$.

[1] Sadahiro Saeki, *A Proof of the Existence of Infinite Product Probability Measures*, Amer. Math. Monthly **103** (1996), no. 8, 682–682.

If $\mathscr{R}$ is a ring of subsets of a set X and $\tau : \mathscr{R} \to [0, \infty]$ is a function such that (i) $\mu(\emptyset) = 0$ and (ii) when $\{A_n\}$ is a countable subset of $\mathscr{R}$ whose members are pairwise disjoint and which satisfies $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$, then

$$\tau\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \tau(A_n),$$

we call τ a **measure on** $\mathscr{R}$. The following is **Hopf's extension theorem**.[2]

Theorem 1 (Hopf's extension theorem)**.** *Suppose that X is a set, that $\mathscr{R}$ is a ring of subsets of X, and that τ is a measure on $\mathscr{R}$. If there is a countable subset $\{E_n\}$ of $\mathscr{R}$ with $\tau(E_n) < \infty$ for each n and such that $\bigcup_{n=1}^{\infty} E_n = X$, then there is a unique measure $\mu : \sigma(\mathscr{R}) \to [0, \infty]$ whose restriction to $\mathscr{R}$ is equal to τ.*

3 Semirings of sets

If X is a set and $\mathscr{S}$ is a collection of subsets of X, we call $\mathscr{S}$ a **semiring of sets** when (i) $\emptyset \in \mathscr{S}$, (ii) if A and B belong to $\mathscr{S}$ then $A \cap B \in \mathscr{S}$, and (iii) if A and B belong to $\mathscr{S}$ then there are pairwise disjoint $C_1, \dots, C_n \in \mathscr{S}$ such that

$$A \setminus B = \bigcup_{i=1}^{n} C_i.$$

If $\mathscr{S}$ is a semiring of subsets of a set X, we call $\mathscr{S}$ a **semialgebra of sets** when $X \in \mathscr{S}$. One proves that if $\mathscr{S}$ is a semialgebra, then the collection $\mathscr{A}$ of all finite unions of elements of $\mathscr{S}$ is equal to the algebra generated by $\mathscr{S}$, and that each element of $\mathscr{A}$ is equal to a finite union of pairwise disjoint elements of $\mathscr{S}$.[3]

4 Cylinder sets

Suppose that $\{(\Omega_i, \mathscr{F}_i, P_i) : i \in I\}$ is a nonempty collection of probability spaces and let

$$\Omega = \prod_{i \in I} \Omega_i.$$

If $A_i \in \mathscr{F}_i$ for each $i \in I$ and $\{i \in I : A_i \neq \Omega_i\}$ is finite, we call

$$A = \prod_{i \in I} A_i$$

a **cylinder set**. Let $\mathscr{C}$ be the collection of all cylinder sets. One checks that $\mathscr{C}$ is a semialgebra of sets.[4]

[2] Karl Stromberg, *Probability for Analysts*, p. 52, Theorem A3.6.
[3] V. I. Bogachev, *Measure Theory*, volume I, p. 8, Lemma 1.2.14.
[4] S. J. Taylor, *Introduction to Measure and Integration*, p. 136, §6.1, Lemma.

Lemma 2. *Suppose that $P : \mathscr{C} \to [0,1]$ is a function such that*

$$\sum_{n=1}^{\infty} P(A_n) = 1$$

whenever A_n are pairwise disjoint elements of $\mathscr{C}$ whose union is equal to Ω. Then there is a unique probability measure on $\sigma(\mathscr{C})$ whose restriction to $\mathscr{C}$ is equal to P.

Proof. Let $\mathscr{A}$ be the collection of all finite unions of cylinder sets. Because $\mathscr{C}$ is a semialgebra of sets, $\mathscr{A}$ is the algebra of sets generated by $\mathscr{C}$, and any element of $\mathscr{A}$ is equal to a finite union of pairwise disjoint elements of $\mathscr{C}$. Let $A \in \mathscr{A}$. There are pairwise disjoint $B_1, \ldots, B_j \in \mathscr{C}$ whose union is equal to A. Suppose also that $\{C_i\}$ is a countable subset of $\mathscr{C}$ with pairwise disjoint members whose union is equal to A. Moreover, as $\Omega \setminus A \in \mathscr{A}$ there are pairwise disjoint $W_1, \ldots, W_p \in \mathscr{C}$ such that $\Omega \setminus A = \bigcup_{i=1}^{p} W_i$. On the one hand, $W_1, \ldots, W_p, B_1, \ldots, B_j$ are pairwise disjoint cylinder sets with union Ω, so

$$\sum_{i=1}^{j} P(B_i) + \sum_{i=1}^{p} P(W_i) = 1.$$

On the other hand, $W_1, \ldots, W_p, C_1, C_2, \ldots$ are pairwise disjoint cylinder sets with union Ω, so

$$\sum_{i=1}^{\infty} P(C_i) + \sum_{i=1}^{p} P(W_i) = 1.$$

Hence,

$$\sum_{i=1}^{j} P(B_i) = \sum_{i=1}^{\infty} P(C_i);$$

this conclusion does not involve $W_1, \ldots, W_p$. Thus it makes sense to define $\tau(A)$ to be this common value, and this defines a function $\tau : \mathscr{A} \to [0,1]$. For $C \in \mathscr{C}$, $\tau(C) = P(C)$, i.e. the restriction of τ to P is equal to $\mathscr{C}$.

If $\{A_n\}$ is a countable subset of $\mathscr{A}$ whose members are pairwise disjoint and $A = \bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$, for each n let $C_{n,1}, \ldots, C_{n,j(n)} \in \mathscr{C}$ be pairwise disjoint cylinder sets with union A_n. Then

$$\{C_{n,i} : n \geq 1, 1 \leq i \leq j(n)\}$$

is a countable subset of $\mathscr{C}$ whose elements are pairwise disjoint and with union A, so

$$\tau(A) = \sum_{n=1}^{\infty} \sum_{i=1}^{j(n)} P(C_{n,i}).$$

But for each n,

$$\tau(A_n) = \sum_{i=1}^{j(n)} P(C_{n,i}),$$

so

$$\tau(A) = \sum_{n=1}^{\infty} \tau(A_n).$$

This shows that $\tau : \mathscr{A} \to [0,1]$ is a measure. Then applying Hopf's extension theorem, we get that there is a unique measure $\mu : \sigma(\mathscr{A}) \to [0,1]$ whose restriction to $\mathscr{A}$ is equal to τ. It is apparent that the σ-algebra generated by a semialgebra is equal to the σ-algebra generated by the algebra generated by the semialgebra, so $\sigma(\mathscr{A}) = \sigma(\mathscr{C})$. Because the restriction of τ to $\mathscr{C}$ is equal to P, the restriction of μ to $\mathscr{C}$ is equal to P. Now suppose that $\nu : \sigma(\mathscr{A}) \to [0,1]$ is a measure whose restriction to $\mathscr{C}$ is equal to P. For $A \in \mathscr{A}$, there are disjoint $C_1, \ldots, C_n \in \mathscr{C}$ with $A = \bigcup_{i=1}^n C_i$. Then

$$\nu(A) = \sum_{i=1}^{n} \nu(C_i) = \sum_{i=1}^{n} P(C_i) = \sum_{i=1}^{n} \mu(C_i) = \mu(A),$$

showing that the restriction of ν to $\mathscr{A}$ is equal to the restriction of μ to $\mathscr{A}$, from which it follows that $\nu = \mu$. ☐

5 Product measures

Suppose that $\{(\Omega_i, \mathscr{F}_i, P_i) : i \in I\}$ is a nonempty collection of probability spaces. The **product σ-algebra** is $\sigma(\mathscr{C})$, the σ-algebra generated by the cylinder sets. We define $P : \mathscr{C} \to [0,1]$ by

$$P(A) = \prod_{i \in I_A} P_i(A_i) = \prod_{i \in I} P_i(A_i),$$

for $A \in \mathscr{C}$ and with $I_A = \{i \in I : A_i \neq \Omega_i\}$, which is finite.

Lemma 3. *Suppose that I is the set of positive integers. If $\{A_n\}$ is a countable subset of $\mathscr{C}$ with pairwise disjoint elements whose union is equal to Ω, then*

$$\sum_{n=1}^{\infty} P(A_n) = 1.$$

Proof. For each $k \geq 1$, there is some i_k and $A_{k,1} \in \mathscr{F}_1, \ldots, A_{k,i_k} \in \mathscr{F}_{i_k}$ such that

$$A_k = \prod_{i=1}^{\infty} A_{k,i},$$

with $A_{k,i} = \Omega_i$ for $i > i_k$. Let $m \geq 1$, let $x = (x_i) \in A_m$, and let $n \geq 1$. If $n = m$,

$$\left(\prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i)\right) \left(\prod_{i > i_m} P_i(A_{n,i})\right) = 1 = \delta_{m,n}.$$

If $m \neq n$ and $y_i \in \Omega_i$ for each $i > i_m$ and we set $y_i = x_i$ for $1 \leq i \leq i_m$, then because A_m and A_n are disjoint and $y \in A_m$, we have $y \notin A_n$ and therefore there is some i, $1 \leq i \leq i_n$, such that $y_i \notin A_{n,i}$. Thus

$$\left(\prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i)\right)\left(\prod_{i>i_m} \chi_{A_{n,i}}(y_i)\right) = \prod_{i=1}^{\infty} \chi_{A_{n,i}}(y_i) = 0. \tag{1}$$

Either $i_n \leq i_m$ or $i_n > i_m$. In the case $i_n \leq i_m$ we have $A_{n,i} = \Omega_i$ for $i > i_m$ and thus

$$\left(\prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i)\right)\left(\prod_{i>i_m} \chi_{A_{n,i}}(y_i)\right) = \prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i),$$

hence by (1),

$$\left(\prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i)\right)\left(\prod_{i>i_m} P_i(A_{n,i})\right) = \prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i) = 0 = \delta_{m,n}.$$

In the case $i_n > i_m$, we have $A_{n,i} = \Omega_i$ for $i > i_n$ and thus

$$\left(\prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i)\right)\left(\prod_{i>i_m} \chi_{A_{n,i}}(y_i)\right) = \left(\prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i)\right)\left(\prod_{i=i_m+1}^{i_n} \chi_{A_{n,i}}(y_i)\right),$$

hence by (1) we have that for $y_i \in \Omega_i$, $i > i_m$,

$$\left(\prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i)\right)\left(\prod_{i=i_m+1}^{i_n} \chi_{A_{n,i}}(y_i)\right) = 0.$$

Therefore, integrating over Ω_i for $i = i_m + 1, \ldots, i_n$,

$$\left(\prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i)\right)\left(\prod_{i=i_m+1}^{i_n} P_i(A_{n,i})\right) = 0,$$

so

$$\left(\prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i)\right)\left(\prod_{i>i_m} P_i(A_{n,i})\right) = 0 = \delta_{m,n}.$$

We have thus established that for any $m \geq 1$, $x \in A_m$, and $n \geq 1$,

$$\left(\prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i)\right)\left(\prod_{i>i_m} P_i(A_{n,i})\right) = \delta_{m,n}. \tag{2}$$

Suppose by contradiction that

$$\sum_{n=1}^{\infty} P(A_n) < 1,$$

i.e.

$$\sum_{n=1}^{\infty}\prod_{i=1}^{\infty}P_i(A_{n,i})<1. \tag{3}$$

If

$$\sum_{n=1}^{\infty}\chi_{A_{n,1}}(x_1)\prod_{i=2}^{\infty}P_i(A_{n,i})=1$$

for all $x_1\in\Omega_1$, then integrating over Ω_1 we contradict (3). Hence there is some $x_1\in\Omega_1$ such that

$$\sum_{n=1}^{\infty}\chi_{A_{n,1}}(x_1)\prod_{i=2}^{\infty}P_i(A_{n,i})<1.$$

Suppose by induction that for some $j\geq 1$, $x_1\in\Omega_1,\ldots,x_j\in\Omega_j$ and

$$\sum_{n=1}^{\infty}\left(\prod_{i=1}^{j}\chi_{A_{n,i}}(x_i)\right)\left(\prod_{i=j+1}^{\infty}P_i(A_{n,i})\right)<1.$$

If

$$\sum_{n=1}^{\infty}\left(\prod_{i=1}^{j+1}\chi_{A_{n,i}}(x_i)\right)\left(\prod_{i=j+2}^{\infty}P_i(A_{n,i})\right)=1$$

for all $x_{j+1}\in\Omega_{j+1}$, then integrating over Ω_{j+1} we contradict (3). Hence there is some $x_{j+1}\in\Omega_{j+1}$ such that

$$\sum_{n=1}^{\infty}\left(\prod_{i=1}^{j+1}\chi_{A_{n,i}}(x_i)\right)\left(\prod_{i=j+2}^{\infty}P_i(A_{n,i})\right)<1.$$

Therefore, by induction we obtain that for any j, there are $x_1\in\Omega_1,\ldots,x_j\in\Omega_j$ such that

$$\sum_{n=1}^{\infty}\left(\prod_{i=1}^{j}\chi_{A_{n,i}}(x_i)\right)\left(\prod_{i=j+1}^{\infty}P_i(A_{n,i})\right)<1. \tag{4}$$

Write $x=(x_1,x_2,\ldots)\in\Omega$. Because $\Omega=\bigcup_{m=1}^{\infty}A_m$, there is some m for which $x\in A_m$. For $j=i_m$, (4) states

$$\sum_{n=1}^{\infty}\left(\prod_{i=1}^{i_m}\chi_{A_{n,i}}(x_i)\right)\left(\prod_{i>i_m}P_i(A_{n,i})\right)<1.$$

But (2) tells us

$$\sum_{n=1}^{\infty}\left(\prod_{i=1}^{i_m}\chi_{A_{n,i}}(x_i)\right)\left(\prod_{i>i_m}P_i(A_{n,i})\right)=\sum_{n=1}^{\infty}\delta_{m,n}=1,$$

a contradiction. Therefore,

$$\sum_{n=1}^{\infty} P(A_n) = 1,$$

proving the claim. □

Lemma 4. *Suppose that I is an uncountable set. If $\{A_n\}$ is a countable subset of $\mathscr{C}$ with pairwise disjoint elements whose union is equal to Ω, then*

$$\sum_{n=1}^{\infty} P(A_n) = 1.$$

Proof. For each n, there are $A_{n,i} \in \mathscr{F}_i$ with $A_{n,i} = \Omega_i$, and $I_n = \{i \in I : A_i \neq \Omega_i\}$ is finite. Then $J = \bigcup_{n=1}^{\infty} I_n$ is countable. Let $\Omega_J = \prod_{i \in J} \Omega_i$, let $\mathscr{C}_J$ be the collection of cylinder sets corresponding to the probability spaces $\{(\Omega_i, \mathscr{F}_i, P_i) : i \in J\}$, and define $P_J : \mathscr{C}_J \to [0,1]$ by

$$P_J(B) = \prod_{i \in J_B} P_i(B_i) = \prod_{i \in J} P_i(B_i),$$

for $B \in \mathscr{C}_J$ and with $J_B = \{i \in J : B_i \neq \Omega_i\}$, which is finite. P_J satisfies

$$P_J(B) = P\left(B \times \prod_{i \in I \setminus J} \Omega_i\right), \qquad B \in \mathscr{C}_J.$$

Let $B_n = \prod_{i \in J} A_{n,i}$, i.e. $A_n = B_n \times \prod_{i \in I \setminus J} A_{n,i}$. Then $\{B_n\}$ is a countable subset of $\mathscr{C}_J$ with pairwise disjoint elements whose union is equal to Ω_J, and applying Lemma 3 we get that

$$\sum_{n=1}^{\infty} P_J(B_n) = 1,$$

and therefore

$$\sum_{n=1}^{\infty} P(A_n) = 1.$$

□

Now by Lemma 2 and the above lemma, there is a unique probability measure μ on $\sigma(\mathscr{C})$ whose restriction to $\mathscr{C}$ is equal to P. That is, when $\{(\Omega_i, \mathscr{F}_i, P_i) : i \in I\}$ are probability spaces and $\mathscr{C}$ is the collection of cylinder sets corresponding to these probability spaces, with $\Omega = \prod_{i \in I} \Omega_i$ and $P : \mathscr{C} \to [0,1]$ defined by

$$P(A) = \prod_{i \in I} P(A_i)$$

for $A = \prod_{i \in I} A_i \in \mathscr{C}$, then there is a unique probability measure μ on the the product σ-algebra such that $\mu(A) = P(A)$ for each cylinder set A. We call μ the **product measure**, and write

$$\bigotimes_{i \in I} \mathscr{F}_i = \sigma(\mathscr{C})$$

and

$$\prod_{i \in I} P_i = \mu.$$