Projection-valued measures and spectral integrals

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April 16, 2014

Abstract

The purpose of these notes is to precisely define all the objects one needs to talk about projection-valued measures and the spectral theorem.

1 Orthogonal complements

Let H be a separable complex Hilbert space.

If W is a subset of H, the *span* of W is the set of all finite linear combinations of elements of W. If W_α, $\alpha \in I$, are subsets of H, let

$$\bigvee_{\alpha \in I} W_\alpha$$

denote the closure of the span of $\bigcup_{\alpha \in I} W_\alpha$. It is straightforward to check that this is the intersection of all closed subspaces of H that contain each of the W_α.[1]

If V is a subspace of H, define

$$V^\perp = \{f \in H : \langle f, g \rangle = 0 \text{ for all } g \in V\},$$

called the *orthogonal complement* of V. If V is a subspace of H then $V^\perp$ is a closed subspace of H.

Let V be a closed subspace of H and let $f \in H$. It is a fact that there is a unique $g_0 \in V$ such that

$$\|f - g_0\| = \inf_{g \in V} \|f - g\|,$$

[1] Let $\mathfrak{L}$ be the set of all closed subspaces of H. Set inclusion $\subseteq$ is a partial order on $\mathfrak{L}$. $\mathfrak{L}$ is a *lattice*: if $M, N \in \mathfrak{L}$, then $M \cap N$ is the greatest element in $\mathfrak{L}$ that is contained both in M and in N, and $M \vee N$ is the least element in $\mathfrak{L}$ that contains each of M and N. $\mathfrak{L}$ is a *complete* lattice: if $M_\alpha \in \mathfrak{L}$, then $\bigcup_\alpha M_\alpha$ is the greatest element in $\mathfrak{L}$ that is contained in each M_α, and $\bigvee_\alpha M_\alpha$ is the least element in $\mathfrak{L}$ that contains each M_α. (For comparison, the set of finite dimensional subspaces of an infinite dimensional Hilbert space is a lattice but is not a complete lattice.)

and that $f - g_0 \in V^\perp$. Then, $f = g_0 + (f - g_0) \in V + V^\perp$. Therefore, if V is a closed subspace of H then

$$H = V + V^\perp. \tag{1}$$

If V and W are subspaces of H, we write $V \perp W$ if $\langle v, w\rangle = 0$ for all $v \in V, w \in W$. If V, W are closed subspaces of H and $V \perp W$, we can show that $V + W = V \vee W$, and thus $V + W$ is in particular a closed subspace of H.[2] By induction, we obtain $V_1 + \cdots + V_n = \bigvee_{k=1}^n V_k$. If V_k, $k \geq 1$, are mutually orthogonal closed subspaces of H, we define

$$\bigoplus_{k=1}^\infty V_k = \bigvee_{k=1}^\infty V_k,$$

called the *orthogonal direct sum* of the subspaces V_k. The orthogonal direct sum $\bigoplus_{k=1}^\infty V_k$ is equal to the closure of the set of all finite sums of the form $\sum v_k$, where $v_k \in V_k$.[3]

If V is a closed subspace of H, then (1) is

$$H = V \oplus V^\perp.$$

The subobjects of a Hilbert space that we care about are mostly closed subspaces, because they are themselves Hilbert spaces (a closed subset of a complete metric space is itself a complete metric space). Generally, when talking about one type of object, we care mostly about those subsets of it that are the same type of object, which is why, for example, that we are so glad to know when closed subspaces are mutually orthogonal, because then their sum is itself a closed subspace.[4]

2 Projections

Let V be a closed subspace of H. The *projection* onto V is the map $P_V : H \to H$ defined in the following way: if $f \in H = V \oplus V^\perp$, let $f = g + h, g \in V, h \in V^\perp$, and set $P_V(f) = g$. The image of P_V is V: if $f \in \operatorname{im} P_V$ then there is some $f_0 \in H$ such that $P_V f_0 = f$ and $f_0 = g + h$, $g \in V, h \in V^\perp$, so by the definition of P_V we get $P_V f_0 = g$, giving $f = g \in V$; on the other hand, if $f \in V$ then $P_V f = f$, so $f \in \operatorname{im} P$.

[2]If $z_n \in V + W$ and $z_n \to z \in H$, write $z_n = v_n + w_n$, $v_n \in V$ and $w_n \in W$. Using $V \perp W$ we get $\|z_n - z_m\|^2 = \|v_n - v_m\|^2 + \|w_n - w_m\|^2$, which we use to prove the claim. On the other hand, there are examples of closed subspaces V, W of a Hilbert space that do not satisfy $V \perp W$ for which $V + W$ is not itself closed.

[3]See Paul Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, p. 26, §13, Theorem 2.

[4]Paul Halmos in his *Introduction to Hilbert Space and the Theory of Spectral Multiplicity* uses the term linear manifold to refer to what I call a subspace and subspace to refer to what I call a closed subspace. I think even better terms would be *linear subspace* and *Hilbert subspace*, respectively; this would emphasize the category of objects with which one is working.

The image of a projection is closed.[5] Hence if P is a projection, we have

$$H = \operatorname{im} P_V \oplus (\operatorname{im} P_V)^\perp = \operatorname{im} P_V \oplus \ker P_V.$$

If P_V is a projection onto a closed subspace V of H, we check that $P_V \in B(H)$. If $g \in V, h \in V^\perp$, then, because $\langle g, h\rangle = 0$,

$$\|P_V(g+h)\|^2 = \|g\|^2 \le \|g\|^2 + \|h\|^2 = \langle g, g\rangle + \langle h, h\rangle = \langle g+h, g+h\rangle = \|g+h\|^2,$$

so $\|P_V\| \le 1$. If $V \neq \{0\}$, let $g \in V$, $g \neq 0$. Then

$$\left\| \frac{g}{\|g\|} \right\| = 1, \qquad \left\| P_V\left(\frac{g}{\|g\|}\right) \right\| = \left\| \frac{g}{\|g\|} \right\| = 1.$$

Hence if P_V is a projection and $P_V \neq 0$, then $\|P_V\| = 1$. Of course, if $P_V = 0$ then $\|P_V\| = 0$. Using $H = V \oplus V^\perp$, we check that if $f_1, f_2 \in H$ then

$$\langle P_V f_1, f_2\rangle = \langle f_1, P_V f_2\rangle,$$

hence P_V is self-adjoint.

We call $P \in B(H)$ *idempotent* if $P^2 = P$.[6] The following conditions are each equivalent to a nonzero idempotent P being a projection:

- P is positive
- P is self-adjoint
- P is normal
- $\ker P = (\operatorname{im} P)^\perp$
- $\|P\| = 1$

3 The lattice of projections

Two important projections: id_H is the projection onto H, and 0 is the projection onto $\{0\}$.

If $T \in B(H)$ is self-adjoint, we say that T is *positive* if, for all $v \in H$,

$$\langle Tv, v\rangle \ge 0.$$

If $S, T \in B(H)$ are self-adjoint, we say that $S \ge T$ if, for all $v \in H$,

$$\langle Sv, v\rangle \ge \langle Tv, v\rangle.$$

[5] We call $P : H \to H$ a projection if there is some closed subspace V of H such that P is the projection onto V, and this V will be the image of P. We often talk about a projection in H without specifying what it is a projection onto.

[6] Often the term projection is used to refer to a thing we would call an idempotent, and the term orthogonal projection is used to refer to a thing we would call a projection.

This is equivalent to $S - T$ being a positive operator. Thus, $T \in B(H)$ is a positive operator if and only if $T \geq 0$. One checks that $\leq$ is a partial order on the set of bounded self-adjoint operators on H.

Let $P \in B(H)$ be a projection. As $H = \operatorname{im} P \oplus (\operatorname{im} P)^\perp = \operatorname{im} P \oplus \ker P$, if $v \in H$ then $v = v_1 + v_2$, $v_1 \in \operatorname{im} P, v_2 \in (\operatorname{im} P)^\perp$,

$$\langle Pv, v\rangle = \langle v_1, v_1 + v_2\rangle = \langle v_1, v_1\rangle + \langle v_1, v_2\rangle = \langle v_1, v_1\rangle \geq 0.$$

Hence $P \geq 0$.

Let $P, Q \in B(H)$ be projections. It is a fact that the following statements are equivalent:

- $P \leq Q$
- $\operatorname{im} P \subseteq \operatorname{im} Q$
- $QP = P$
- $PQ = P$
- $\|Pv\| \leq \|Qv\|$ for all $v \in H$

The set of projections is a complete lattice.[7]

4 Sums of projections

If $P, Q \in B(H)$ are projections, we write $P \perp Q$ if $\operatorname{im} P \perp \operatorname{im} Q$, which is equivalent to $PQ = 0$ and $QP = 0$; neither one of those by itself is sufficient. It is a fact[8] that $P + Q$ is a projection if and only if $P \perp Q$, in which case $P + Q$ is a projection onto $\operatorname{im} P \oplus \operatorname{im} Q$. The product PQ is a projection if and only if $PQ = QP$, in which case it is a projection onto $\operatorname{im} P \cap \operatorname{im} Q$.

Paul Halmos shows the following in Question 94 of his *Hilbert Space Problem Book*: If $T_n \in B(H)$ are self-adjoint such that $T_n \leq T_{n+1}$ for all n and if there is some self-adjoint $T' \in B(H)$ such that $T_n \leq T'$ for all n, then there is some self-adjoint $T \in B(H)$ such that $T_n \to T$ in the *strong operator topology*.[9] This is analogous to how a nondecreasing sequence of real numbers that has an upper bound converges to a real number.

Let M_n, $n \geq 1$, be closed subspaces of H such that $M_n \subseteq M_{n+1}$ and let P_n be the projection onto M_n. As $M_n \subseteq M_{n+1}$ we have $P_n \leq P_{n+1}$ for all n. Also, $P_n \leq \mathrm{id}_H$ for all n, as id_H is the projection onto H and $M_n \subseteq H$. Hence by

[7] If $P, Q \in B(H)$ are projections, then the infimum of P and Q is the projection onto $\operatorname{im} P \cap \operatorname{im} Q$, and the supremum of P and Q is the projection onto $\operatorname{im} P \vee \operatorname{im} Q$. See Problem 96 of Paul Halmos's *Hilbert Space Problem Book*.

[8] See Steven Roman, *Advanced Linear Algebra*, third ed., p. 78

[9] Recall that if $T_n \in B(H)$ and $T \in B(H)$, we say that $T_n \to T$ in the strong operator topology if for all $v \in H$ we have $T_n v \to Tv$. (If $T_n \to T$ in $B(H)$ then $T_n \to T$ in the strong operator topology.)

the result from Halmos stated above, there is some self-adjoint $P \in B(H)$ such that $P_n \to P$ in the strong operator topology. Let

$$M = \bigvee_{n=1}^{\infty} M_n.$$

I claim that P is the projection onto M.[10]

If $P_n \in B(H)$, $n \geq 1$, are projections and $P_i \perp P_j$ for $i \neq j$, let

$$M_n = \bigoplus_{k=1}^{n} \operatorname{im} P_k.$$

Define $T_n v = \sum_{k=1}^{n} P_k v$. As M_n is a closed subspace of H, we have

$$H = M_n \oplus M_n^{\perp} = \operatorname{im} P_1 \oplus \cdots P_n \oplus M_n^{\perp}.$$

If $v = v_1 + \cdots + v_n + v'$, $v_1 \in \operatorname{im} P_1, \ldots, v_n \in \operatorname{im} P_n, v' \in M_n^{\perp}$, then

$$T_n v = \sum_{k=1}^{n} (P_k(v_1 + \cdots + v_n + v')) = \sum_{k=1}^{n} v_k.$$

Thus T_n is the projection onto M_n. Therefore, there is some self-adjoint $P \in B(H)$ such that $T_n \to P$ in the strong operator topology, and with

$$M = \bigvee_{n=1}^{\infty} \bigvee_{k=1}^{n} \operatorname{im} P_k = \bigvee_{n=1}^{\infty} \operatorname{im} P_k = \bigoplus_{n=1}^{\infty} \operatorname{im} P_n,$$

P is the projection onto M. $\sum_{k=1}^{n} P_k \to P$ in the strong operator topology, and we denote P by $\sum_{k=1}^{\infty} P_k$.

5 Definition of projection-valued measures

Let $\mathscr{P}(H)$ be the set of projections in $B(H)$. Let $\mathscr{B}(\mathbb{C})$ be the Borel σ-algebra of $\mathbb{C}$. A *projection-valued measure* on $\mathbb{C}$ is a map $E : \mathscr{B}(\mathbb{C}) \to \mathscr{P}(H)$ such that

- $E(\emptyset) = 0$ and $E(\mathbb{C}) = \mathrm{id}_H$
- If $B_n \in \mathscr{B}(\mathbb{C})$, $n \geq 1$, are pairwise disjoint, then

  $$E\Big(\bigcup_{n\geq 1} B_n\Big) = \sum_{n \geq 1} E(B_n),$$

  where $\sum_{n\geq 1} E(B_n)$ is the limit of $\sum_{1 \leq n \leq N} E(B_n)$ in the strong operator topology.

[10]cf. Paul Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, p. 46, §28, Theorem 1.

6 Finite additivity

If $E : \mathscr{B}(\mathbb{C}) \to \mathscr{P}(H)$ is any function that satisfies $E(B_1 \cup B_2) = E(B_1) + E(B_2)$ for disjoint $B_1, B_2 \in \mathscr{B}(\mathbb{C})$, then it satisfies the following four properties. In particular, if E is a projection-valued measure it satisfies them.

1. $E(\emptyset) = E(\emptyset \cup \emptyset) = E(\emptyset) + E(\emptyset)$, so $E(\emptyset) = 0$.

2. If $B_1, B_2 \in \mathscr{B}(\mathbb{C})$ and $B_1 \subseteq B_2$, then

$$E(B_2) = E(B_2 \setminus B_1 \cup B_1) = E(B_2 \setminus B_1) + E(B_1) \geq E(B_1),$$

since $E(B_2 \setminus B_1)$ is a projection and hence is a positive operator. Therefore, if $B_1 \subseteq B_2$ then $E(B_1) \leq E(B_2)$.

3. Let $B_1, B_2 \in \mathscr{B}(\mathbb{C})$. We have

$$E(B_1) = E(B_1 \cap B_2) + E(B_1 \setminus B_2),$$

and

$$E(B_2) = E(B_1 \cap B_2) + E(B_2 \setminus B_1),$$

and

$$E(B_1 \cup B_2) = E(B_1 \cap B_2) + E(B_1 \setminus B_2) + E(B_2 \setminus B_1),$$

and combining these gives, for any $B_1, B_2 \in \mathscr{B}(\mathbb{C})$,

$$\begin{aligned} E(B_1) + E(B_2) &= E(B_1 \cap B_2) + E(B_1 \cap B_2) + E(B_1 \setminus B_2) + E(B_2 \setminus B_1) \\ &= E(B_1 \cap B_2) + E(B_1 \cup B_2). \end{aligned}$$

4. Let $B_1, B_2 \in \mathscr{B}(\mathbb{C})$. Multiplying both sides of the above equation on the left by $E(B_2)$ gives

$$E(B_1)E(B_2) + E(B_2)E(B_2) = E(B_1 \cap B_2)E(B_2) + E(B_1 \cup B_2)E(B_2).$$

As $E(B_1 \cup B_2)$ and $E(B_2)$ are projections and $E(B_1 \cup B_2) \geq E(B_2)$, we have

$$E(B_1 \cup B_2)E(B_2) = E(B_2),$$

which with $E(B_2)E(B_2) = E(B_2)$ gives

$$E(B_1)E(B_2) + E(B_2) = E(B_1 \cap B_2) + E(B_2).$$

Hence, for any $B_1, B_2 \in \mathscr{B}(\mathbb{C})$,

$$E(B_1)E(B_2) = E(B_1 \cap B_2).$$

7 Complex measures

Suppose that $E : \mathscr{B}(\mathbb{C}) \to \mathscr{P}(H)$ is a function such that $E(\mathbb{C}) = \mathrm{id}_H$, and that for all $v, w \in H$, the function $E_{v,w} : \mathscr{B}(\mathbb{C}) \to \mathbb{C}$ defined by

$$E_{v,w}(B) = \langle E(B)v, w \rangle$$

is a complex measure. I will show that E is a projection-valued measure. Let $B_n \in \mathscr{B}(\mathbb{C})$, $n \geq 1$, be pairwise disjoint.[11] If $B_1, B_2 \in \mathscr{B}(\mathbb{C})$ are disjoint, then for all $v, w \in H$,

$$\begin{aligned} \langle E(B_1 \cup B_2)v, w \rangle &= E_{v,w}(B_1 \cup B_2) \\ &= E_{v,w}(B_1) + E_{v,w}(B_2) \\ &= \langle E(B_1)v, w \rangle + \langle E(B_2)v, w \rangle \\ &= \langle (E(B_1) + E(B_2))v, w \rangle , \end{aligned}$$

and since this holds for all $v, w \in H$, we obtain $E(B_1 \cup B_2) = E(B_1) + E(B_2)$. Therefore, from §6, if $B_1, B_2 \in \mathscr{B}(\mathbb{C})$ are disjoint then

$$E(B_1)E(B_2) = E(B_1 \cap B_2) = E(\emptyset) = 0.$$

If $v \in H$ and $v_n = E(B_n)v$, then, for $m \neq n$,

$$\langle v_n, v_m \rangle = \langle E(B_n)v, E(B_m)v \rangle = \langle E(B_m)E(B_n)v, v \rangle = \langle 0v, 0 \rangle = 0.$$

For $a_n \in \mathbb{C}$, for the sequence $\sum_{n=1}^{N} a_n v_n$ to converge in H it is equivalent to $\sum_{n=1}^{\infty} \|a_n v_n\|^2 < \infty$;[12] but $\|v_n\| \leq \|v\|$, so it suffices to show that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Using that if T is a projection then T is self-adjoint and $T^2 = T$, and that

[11] cf. Paul Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, p. 59, §36, Theorem 3.

[12] e.g. Walter Rudin's *Functional Analysis*, p. 295, Theorem 12.6: if $x_n \in H$ are pairwise orthogonal, not necessarily of unit norm, then for $\sum_{n=1}^{\infty} x_n$ to converge is equivalent to $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$.

$E_{v,v}$ is a complex measure,

$$\begin{aligned}
\sum_{n=1}^{\infty} \|E(B_n)v\|^2 &= \sum_{n=1}^{N} \langle E(B_n)v, E(B_n)v \rangle \\
&= \sum_{n=1}^{\infty} \langle E(B_n)E(B_n)v, v \rangle \\
&= \sum_{n=1}^{\infty} \langle E(B_n)v, v \rangle \\
&= \sum_{n=1}^{\infty} E_{v,v}(B_n) \\
&= E_{v,v}\left(\bigcup_{n=1}^{\infty} B_n\right) \\
&= \left\langle E\left(\bigcup_{n=1}^{\infty} B_n\right) v, v \right\rangle \\
&= \left\| E\left(\bigcup_{n=1}^{\infty} B_n\right) v \right\|^2 \\
&\leq \|v\|.
\end{aligned}$$

Therefore, the sequence $\sum_{n=1}^{N} E(B_n)v$ converges in H; namely, $\sum_{n=1}^{N} E(B_n)$ converges in the strong operator topology. Let P be its limit. By §4, P is the projection onto $\bigoplus_{n=1}^{\infty} \operatorname{im} E(B_n)$.

For $v, w \in H$,

$$\begin{aligned}
\left\langle E\left(\bigcup_{n=1}^{\infty} B_n\right) v, w \right\rangle &= E_{v,w}\left(\bigcup_{n=1}^{\infty} B_n\right) \\
&= \sum_{n=1}^{\infty} E_{v,w}(B_n) \\
&= \sum_{n=1}^{\infty} \langle E(B_n)v, w \rangle \\
&= \langle Pv, w \rangle,
\end{aligned}$$

and since this is true for all $v, w \in H$, we obtain

$$E\left(\bigcup_{n=1}^{\infty} B_n\right) = P,$$

where P is the limit of $\sum_{n=1}^{N} E(B_n)$ in the strong operator topology. This completes the proof that E is a projection-valued measure.

On the other hand, if $E : \mathscr{B}(\mathbb{C}) \to \mathscr{P}(H)$ is a projection-valued measure, we can show that for each $v, w \in H$ the function $E_{v,w} : \mathscr{B}(\mathbb{C}) \to \mathbb{C}$ defined by $E_{v,w}(B) = \langle E(B)v, w\rangle$ is a complex measure.

8 Spectral integrals

Let $\mathfrak{B}(\mathbb{C})$ be the set of bounded measurable functions $\mathbb{C} \to \mathbb{C}$.[13] It is a complex vector space, and we define the norm $\|f\| = \sup_{z\in\mathbb{C}} |f(z)|$; one checks that $\mathfrak{B}(\mathbb{C})$ is a Banach space. Paul Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, p. 60, §37, proves[14] that if $E : \mathscr{B}(\mathbb{C}) \to \mathscr{P}(H)$ is a projection-valued measure and $f \in \mathfrak{B}$, then there is a unique $A \in B(H)$ such that, for all $v, w \in H$,

$$\langle Av, w\rangle = E_{v,w}(f) = \int f dE_{v,w} = \int_{\mathbb{C}} f(\lambda) dE_{v,w}(\lambda),$$

and $\|A\| \leq 2\|f\|$. We write

$$A = E(f) = \int f dE = \int_{\mathbb{C}} f(\lambda) dE(\lambda).$$

We can check that $\mathfrak{B}(\mathbb{C})$ is a C^*-algebra, with f^* defined by $f^*(z) = \overline{f(z)}$. It is a fact that $B(H)$ is a C^*-algebra. If $\alpha \in \mathbb{C}$ and $f, g \in \mathfrak{B}(\mathbb{C})$, then[15]

$$E(\alpha f) = \alpha E(f), \qquad E(f+g) = E(f) + E(g); \qquad E(f^*) = (E(f))^*. \tag{2}$$

The first two of these together with $\|E(f)\| \leq 2\|f\|$ show that $E : \mathfrak{B}(\mathbb{C}) \to B(H)$ is a bounded linear map. If $f, g \in \mathfrak{B}(\mathbb{C})$, then[16]

$$E(f)E(g) = E(fg).$$

[13] This is not $L^\infty(\mathbb{C})$, the set of equivalence classes of essentially bounded measurable functions, where two functions are equivalent if they are equal almost everywhere. Moreover, it is not even the set $\mathscr{L}^\infty(\mathbb{C})$ of essentially bounded measurable functions.

When does one speak about $\mathscr{L}^\infty(\mathbb{C})$? $\mathscr{L}^\infty(\mathbb{C})$ is a vector space; let $\mathscr{N}(\mathbb{C})$ be the set of those functions that are equal to 0 almost everywhere in $\mathbb{C}$; $\mathscr{N}(\mathbb{C})$ is a vector space; then $L^\infty(\mathbb{C})$ is the vector space quotient $\mathscr{L}^\infty(\mathbb{C})/\mathscr{N}(\mathbb{C})$.

[14] The operator A is the operator obtained from the following statement, which itself follows from the Riesz representation theorem. If $\phi : H \times H \to \mathbb{C}$ is sesquilinear (we take sesquilinear to mean linear in the first entry and conjugate linear in the second entry) and

$$M = \sup_{\|v\|=\|w\|=1} |\phi(v, w)| < \infty,$$

then there exists a unique $A \in B(H)$ such that

$$\phi(v, w) = \langle Av, w\rangle, \qquad v, w \in H,$$

and $\|A\| = M$.

One also has to prove that for each $f \in \mathfrak{B}$, $\phi(v, w) = E_{v,w}(f)$ is sesquilinear.

[15] See Paul Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, p. 60, §37, Theorem 2.

[16] From Paul Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, p. 61, §37, Theorem 3. To understand the proof by Halmos (which is symbolically convincing because of our familiarity with the permissible moves one can make when integrating functions

This and the third statement in (2) show that $E : \mathfrak{B}(\mathbb{C}) \to B(H)$ is a homomorphism of C^*-algebras:

From the fact that $E : \mathfrak{B}(\mathbb{C}) \to B(H)$ is a homomorphism of C^*-algebras, it follows in particular that $E(f)$ is a normal operator for each $f \in \mathfrak{B}(\mathbb{C})$.

9 The spectrum of a projection-valued measure

If $E : \mathscr{B}(\mathbb{C}) \to \mathscr{P}(H)$ is a projection-valued measure, let $U_\alpha, \alpha \in I$, be those open sets $U_\alpha \subseteq \mathbb{C}$ such that $E(U_\alpha) = 0$. The *spectrum* of E is

$$\sigma(E) = \mathbb{C} \setminus \bigcup_{\alpha \in I} U_\alpha;$$

this may also be called the support of E, and is analogous to the support of a nonnegative measure. Since $\sigma(E)$ is the complement of a union of open sets, it is closed.

For each $n \geq 1$, let $D_n = \{|z| \leq n\}$. D_n is compact, so there are $\alpha_{n,1}, \ldots, \alpha_{n,N}$ such that

$$D_n \subseteq \bigcup_{k=1}^{N} U_{\alpha_{n,k}}.$$

But, using §6 and the fact that $E(U_{\alpha_{n,k}}) = 0$ for $1 \leq k \leq N$,

$$\begin{aligned}
E\left(\bigcup_{k=1}^{N} U_{\alpha_{n,k}}\right) &= E(U_{\alpha_{n,1}}) + E\left(\bigcup_{k=2}^{N} U_{\alpha_{n,k}}\right) - E\left(U_{\alpha_{n,1}} \cap \bigcup_{k=2}^{N} U_{\alpha_{n,k}}\right) \\
&= E\left(\bigcup_{k=2}^{N} U_{\alpha_{n,k}}\right) - E(U_{\alpha_{n,1}}) E\left(\bigcup_{k=2}^{N} U_{\alpha_{n,k}}\right) \\
&= E\left(\bigcup_{k=2}^{N} U_{\alpha_{n,k}}\right) \\
&= \cdots \\
&= 0,
\end{aligned}$$

therefore $E(D_n) = 0$ (it will be ≤ 0, and as a projection is a positive operator it must equal 0). Let $B_n = D_{n+1} \setminus D_n$; as B_n is a subset of D_{n+1} we get $E(B_n) = 0$. We have $\bigcup_{n=1}^{\infty} B_n = \mathbb{C}$, and as the B_n are pairwise disjoint we get

$$E(\mathbb{C}) = E\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} E(B_n) = 0,$$

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using complex measures), keep in mind that if $B_1, B_2 \in \mathscr{B}(\mathbb{C})$ then

$$E_{E(B_1)v,w}(B_2) = \langle E(B_2)E(B_1)v, w\rangle = \langle E(B_1 \cap B_2)v, w\rangle = E_{v,w}(B_1 \cap B_2).$$

contradicting that $E(\mathbb{C}) = \mathrm{id}_H$. Therefore, if E is a projection-valued measure then its spectrum is not empty.

Here are some facts about projection-valued measures. $E(\mathbb{C} \setminus \sigma(E)) = 0$.[17]

Let $\mathfrak{B}_E(\mathbb{C})$ be the set of bounded measurable functions $\sigma(E) \to \mathbb{C}$, and let $\|f\|_E = \sup_{z \in \sigma(E)} |f(z)|$. If E is a projection-valued measure with compact spectrum and $f : \mathbb{C} \to \mathbb{C}$ is continuous, then[18]

$$\left\| E(\chi_{\sigma(E)} f) \right\| = \|f\|_E .$$

(We often talk about projection-valued measures whose spectrum is compact; since their spectrum is closed, to demand that the spectrum of a projection-valued measure is compact is to demand that it is bounded.)

If E is a projection-valued measure with compact spectrum and if $A = E(\chi_{\sigma(E)}\lambda)$, then[19]

$$\sigma(A) = \sigma(E).$$

10 Statement of the spectral theorem

The spectral theorem, proved by in Paul Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, p. 69, §43, Theorem 1, states the following:

If $A \in B(H)$ is self-adjoint, then there exists a unique projection-valued measure $E : \mathscr{B}(\mathbb{C}) \to \mathscr{P}(H)$, with $\sigma(E)$ compact and $\sigma(E) \subset \mathbb{R}$, such that

$$A = E(\chi_{\sigma(E)}\lambda) = \int_{\sigma(E)} \lambda dE(\lambda).$$

Since $\sigma(E) \subset \mathbb{R}$, we can write $E : \mathscr{B}(\mathbb{R}) \to \mathscr{P}(H)$.

[17] Paul Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, p. 62, §38, Theorem 1. His statement is about regular measures, but a projection-valued measure on the Borel σ-algebra of $\mathbb{C}$ is regular, as he shows on the page after that.

[18] Paul Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, p. 62, §38, Theorem 2.

[19] Paul Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, p. 64, §39, Theorem 2. The proof uses the fact that $T \in B(H)$ is invertible if and only if there is some $\alpha > 0$ such that $\|Tv\| \geq \alpha \|v\|$ for all $v \in H$.