Regulated functions and the regulated integral

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1 Regulated functions and step functions

Let $I = [a, b]$ and let X be a normed space. A function $f : I \to X$ is said to be *regulated* if for all $t \in [a, b)$ the limit $\lim_{s \to t^+} f(s)$ exists and for all $t \in (a, b]$ the limit $\lim_{s \to t^-} f(s)$ exists. We denote these limits respectively by $f(t^+)$ and $f(t^-)$. We define $R(I, X)$ to be the set of regulated functions $I \to X$. It is apparent that $R(I, X)$ is a vector space. One checks that a regulated function is bounded, and that $R(I, X)$ is a normed space with the norm $\|f\|_\infty = \sup_{t \in [a,b]} \|f(t)\|$.

Theorem 1. *If I is a compact interval in $\mathbb{R}$ and X is a normed algebra, then $R(I, X)$ is a normed algebra.*

Proof. If $f, g \in R(I, X)$, then $fg \in R(I, X)$ because the limit of a product is equal to a product of limits. For $t \in I$ we have

$$\|(fg)(t)\| = \|f(t)g(t)\| \leq \|f(t)\| \, \|g(t)\| \leq \|f\|_\infty \, \|g\|_\infty ,$$

so $\|fg\|_\infty \leq \|f\|_\infty \, \|g\|_\infty$. □

A function $f : I \to X$, where $I = [a, b]$, is said to be a *step function* if there are $a = s_0 < s_1 < \cdots < s_k = b$ for which f is constant on each open interval (s_{i-1}, s_i). We denote the set of step functions $I \to X$ by $S(I, X)$. It is apparent that $S(I, X)$ is contained in $R(I, X)$ and is a vector subspace, and the following theorem states that if X is a Banach space then $S(I, X)$ is dense in $R(I, X)$.[1]

Theorem 2. *Let I be a compact interval in $\mathbb{R}$, let X be a Banach space, and let $f \in X^I$. $f \in R(I, X)$ if and only if for all $\epsilon > 0$ there is some $g \in S(I, X)$ such that $\|f - g\|_\infty < \epsilon$.*

We prove in the following theorem that the set of regulated functions from a compact interval to a Banach space is itself a Banach space.

[1] Jean Dieudonné, *Foundations of Modern Analysis*, enlarged and corrected printing, p. 145, Theorem 7.6.1; Rodney Coleman, *Calculus on Normed Vector Spaces*, p. 70, Proposition 3.3; cf. Robert G. Bartle, *A Modern Theory of Integration*, p. 49, Theorem 3.17.

Theorem 3. *If I is a compact interval in $\mathbb{R}$ and X is a Banach space, then $R(I,X)$ is a Banach space.*

Proof. Let $f_n \in R(I,X)$ be a Cauchy sequence. For each $t \in I$ we have

$$\|f_n(t) - f_m(t)\| \leq \|f_n - f_m\|_\infty ,$$

hence $f_n(t)$ is a Cauchy sequence in X. As X is a Banach space, this Cauchy sequence converges to some limit, and we define $f(t)$ to be this limit. Thus $f \in X^I$ and $\|f - f_n\|_\infty \to 0$. We have to prove that $f \in R(I,X)$. Let $\epsilon > 0$. There is some N for which $n \geq N$ implies that $\|f - f_n\|_\infty < \epsilon$; in particular, $\|f - f_N\|_\infty < \epsilon$. By Theorem 2, there is some $g_N \in S(I,X)$ with $\|f_N - g_N\|_\infty < \epsilon$. Then,

$$\|f - g_N\|_\infty \leq \|f - f_N\|_\infty + \|f_N - g_N\|_\infty < 2\epsilon,$$

and by Theorem 2 this implies that $f \in R(I,X)$. □

The following lemma shows that the set of points of discontinuity of a regulated function taking values in a Banach space is countable.

Lemma 4. *If I is a compact interval in $\mathbb{R}$, X is a Banach space, and $f \in R(I,X)$, then*

$$\{t \in I : f \text{ is discontinuous at } t\}$$

is countable.

Proof. For each n let $g_n \in S(I,X)$ satisfy $\|f - g_n\| \leq \frac{1}{n}$, and let

$$D_n = \{t \in I : g_n \text{ is discontinuous at } t\}.$$

g_n is a step function so D_n is finite, and hence $D = \bigcup_{n=1}^\infty D_n$ is countable. It need not be true that f is discontinuous at each point in D, but we shall prove that if $t \in I \setminus D$ then f is continuous at t, which will prove the claim.

Suppose that $t \in I \setminus D$, let $\epsilon > 0$, and take $N > \frac{1}{\epsilon}$. As $t \notin D_N$, the step function g_N is continuous at t, and hence there is some $\delta > 0$ for which $|s-t| < \delta$ implies that $\|g_N(s) - g_N(t)\| < \epsilon$. If $|s-t| < \delta$, then

$$\begin{aligned}
\|f(s) - f(t)\| &\leq \|f(s) - g_N(s)\| + \|g_N(s) - g_N(t)\| + \|g_N(t) - f(t)\| \\
&\leq 2\|f - g_N\|_\infty + \|g_N(s) - g_N(t)\| \\
&< \frac{2}{N} + \epsilon \\
&< 3\epsilon,
\end{aligned}$$

showing that f is continuous at t. □

2 Integrals of step functions

Let $I = [a, b]$ and let X be a normed space. If $f \in S(I, X)$ then there is a subdivision $a = s_0 < s_1 < \cdots < s_k = b$ of $[a, b]$ and there are $c_i \in X$ such that f takes the value c_i on the open interval (s_{i-1}, s_i). Suppose that there is a subdivision $a = t_0 < t_1 < \cdots < t_l = b$ of $[a, b]$ and $d_i \in X$ such that f takes the value d_i on the open interval (t_{i-1}, t_i). One checks that

$$\sum_{i=1}^{k}(s_i - s_{i-1})c_i = \sum_{i=1}^{l}(t_i - t_{i-1})d_i.$$

We define the *integral* of f to be the above element of X, and denote this element of X by $\int_I f = \int_a^b f$.

Lemma 5. *If I is a compact interval in $\mathbb{R}$ and X is a normed space, then $\int_I : S(I, X) \to X$ is linear.*

Lemma 6. *If $I = [a, b]$ and X is a normed space, then $\int_I : S(I, X) \to X$ is a bounded linear map with operator norm $b - a$.*

Proof. If $f \in S(I, X)$, let $a = s_0 < s_1 < \cdots < s_k = b$ be a subdivision of $[a, b]$ and let $c_i \in X$ such that f takes the value c_i on the open interval (s_{i-1}, s_i). Then,

$$\left\| \int_I f \right\| \leq \sum_{i=1}^{k}(s_i - s_{i-1}) \|c_i\| \leq \sum_{i=1}^{k}(s_i - s_{i-1}) \|f\|_\infty = (b - a) \|f\|_\infty .$$

This shows that $\left\| \int_I \right\| \leq b - a$, and if f is constant, say $f(t) = c \in X$ for all $t \in I$, then $\int_I f = (b - a)c$ and $\left\| \int_I f \right\| = (b - a) \|c\| = (b - a) \|f\|_\infty$, showing that $\left\| \int_I \right\| = b - a$. □

Lemma 7. *If $a \leq b \leq c$, if X is a normed space, and if $g \in S([a, c], X)$, then*

$$\int_a^c g = \int_a^b g + \int_b^c g.$$

3 The regulated integral

Let I be a compact interval in $\mathbb{R}$ and let X be a Banach space. Theorem 2 shows that $S(I, X)$ is a dense subspace of $R(I, X)$, and therefore if $T_0 \in \mathscr{B}(S(I, X), X)$ then there is one and only one $T \in \mathscr{B}(R(I, X), X)$ whose restriction to $S(I, X)$ is equal to T_0, and this operator satisfies $\|T\| = \|T_0\|$. Lemma 6 shows that $\int_I : S(I, X) \to X$ is a bounded linear operator, thus there is one and only one bounded linear operator $R(I, X) \to X$ whose restriction to $S(I, X)$ is equal to $\int_I$, and we denote this operator $R(I, X) \to X$ also by $\int_I$. With $I = [a, b]$, we have $\left\| \int_I \right\| = b - a$. We call $\int_I : R(I, X) \to X$ the *regulated integral*.

Lemma 8. *If $a \le b \le c$, if X is a Banach space, and if $f \in R([a,c],X)$, then*

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

Proof. Let $I_1 = [a,b]$, $I_2 = [b,c]$, $I = [a,c]$, and let f_1 and f_2 be the restriction of f to I_1 and I_2 respectively. From the definition of regulated functions, $f_1 \in R(I_1,X)$ and $f_2 \in R(I_2,X)$. By Theorem 2, for any $\epsilon > 0$ there is some $g \in S(I,X)$ satisfying $\|f-g\|_\infty < \epsilon$. Taking g_1 and g_2 to be the restriction of g to I_1 and I_2, we check that $g_1 \in S(I_1,X)$ and $g_2 \in S(I_2,X)$. Then by Lemma 7,

$$\begin{aligned}
\left\|\int_I f - \int_{I_1} f_1 - \int_{I_2} f_2\right\|_\infty &\le \left\|\int_I f - \int_I g\right\|_\infty + \left\|\int_I g - \int_{I_1} g_1 - \int_{I_2} g_2\right\|_\infty \\
&\quad + \left\|\int_{I_1} g_1 + \int_{I_2} g_2 - \int_{I_1} f_1 - \int_{I_2} f_2\right\|_\infty \\
&= \left\|\int_I (f-g)\right\|_\infty + 0 \\
&\quad + \left\|\int_{I_1} (g_1 - f_1)\right\|_\infty + \left\|\int_{I_2} (g_2 - f_2)\right\|_\infty \\
&\le (c-a)\|f-g\|_\infty + (b-a)\|g_1 - f_2\|_\infty \\
&\quad + (c-b)\|g_2 - f_2\|_\infty .
\end{aligned}$$

But $\|g_1 - f_1\|_\infty \le \|g-f\|_\infty$ and $\|g_2 - f_2\|_\infty \le \|g-f\|_\infty$, hence we obtain

$$\left\|\int_I f - \int_{I_1} f_1 - \int_{I_2} f_2\right\|_\infty < (c-a)\epsilon + (b-a)\epsilon + (c-b)\epsilon = 2(c-a)\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we get

$$\left\|\int_I f - \int_{I_1} f_1 - \int_{I_2} f_2\right\|_\infty = 0,$$

so

$$\int_I f = \int_{I_1} f_1 + \int_{I_2} f_2,$$

proving the claim. □

We prove that applying a bounded linear map and taking the regulated integral commute.[2]

Lemma 9. *Suppose that I is a compact interval in $\mathbb{R}$ and that X and Y are Banach spaces. If $f \in R(I,X)$ and $T \in \mathscr{B}(X,Y)$, then $T \circ f \in R(I,Y)$ and*

$$\int_I T \circ f = T \int_I f.$$

[2] Jean-Paul Penot, *Calculus Without Derivatives*, p. 124, Proposition 2.18.

Proof. Because T is continuous we have $T \circ f \in R(I, Y)$. For $\epsilon > 0$, there is some $g \in S(I, X)$ satisfying $\|f - g\|_\infty < \epsilon$. Write $I = [a, b]$. Because g is a step function, there is a subdivision $a = s_0 < s_1 < \cdots < s_k = b$ of I and there are $c_i \in X$ such that g takes the value c_i on the open interval (s_{i-1}, s_i). Furthermore, $T \circ g$ takes the value Tc_i on the open interval (s_{i-1}, s_i) so $T \circ g \in S(I, Y)$, and then because T is linear,

$$\int_I T \circ g = \sum_{i=1}^{k} (s_i - s_{i-1}) T c_i = T \sum_{i=1}^{k} (s_i - s_{i-1}) c_i = T \int_I g.$$

Using this,

$$\begin{aligned}
\left\| \int_I T \circ f - T \int_I f \right\| &\leq \left\| \int_I T \circ f - \int_I T \circ g \right\| + \left\| \int_I T \circ g - T \int_I g \right\| \\
&\quad + \left\| T \int_I g - T \int_I f \right\| \\
&= \left\| \int_I T \circ (f - g) \right\| + \left\| T \int_I (f - g) \right\| \\
&\leq (b - a) \|T \circ (f - g)\|_\infty + \|T\| \left\| \int_I (f - g) \right\| \\
&\leq (b - a) \|T\| \|f - g\|_\infty + \|T\| (b - a) \|f - g\|_\infty \\
&< 2(b - a) \|T\| \epsilon.
\end{aligned}$$

As $\epsilon > 0$ is arbitrary, this means that

$$\left\| \int_I T \circ f - T \int_I f \right\| = 0,$$

and so

$$\int_I T \circ f = T \int_I f.$$

□

4 Left and right derivatives

Suppose that I is an open interval in $\mathbb{R}$, X is a normed space, $f \in X^I$, and $t \in I$. We say that f is *right-differentiable at* t if $\frac{f(t+h) - f(t)}{h}$ has a limit as $h \to 0^+$, and that f is *left-differentiable at* t if $\frac{f(t+h) - f(t)}{h}$ has a limit as $h \to 0^-$. We call these limits respectively the *right derivative of* f *at* t and the *left derivative of* f *at* t, denoted respectively by $f'_+(t)$ and $f'_-(t)$. For f to be differentiable at t means that $f'_+(t)$ and $f'_-(t)$ exist and are equal.

The following is the mean value theorem for functions taking values in a Banach space.[3]

[3] Henri Cartan, *Differential Calculus*, p. 39, Theorem 3.1.3.

Theorem 10 (Mean value theorem)**.** *Suppose that $I = [a, b]$, that X is a Banach space, and that $f : I \to X$ and $g : I \to \mathbb{R}$ are continuous functions. If there is a countable set $D \subset I$ such that $t \in I \setminus D$ implies that $f'_+(t)$ and $g'_+(t)$ exist and satisfy $\left\| f'_+(t) \right\| \leq g'_+(t)$, then*

$$\| f(b) - f(a) \| \leq g(b) - g(a).$$

Corollary 11. *Suppose that $I = [a, b]$, that X is a Banach space, and that $f : I \to X$ is continuous. If there is a countable set $D \subset I$ such that $t \in I \setminus D$ implies that $f'_+(t) = 0$, then f is constant on I.*

5 Primitives

Let $I = [a, b]$, let X be a normed space, and let $f, g \in X^I$. We say that g is a *primitive of f* if g is continuous and if there is a countable set $D \subset I$ such that $t \in I \setminus D$ implies that g is differentiable at t and $g'(t) = f(t)$.

Lemma 12. *Suppose that I is a compact interval in $\mathbb{R}$, that X is a Banach space, and that $f : I \to X$ is a function. If $g_1, g_2 : I \to X$ are primitives of f, then $g_1 - g_2$ is constant on I.*

Proof. For $i = 1, 2$, as g_i is a primitive of f there is a countable set $D_i \subset I$ such that $t \in I \setminus D_i$ implies that g_i is differentiable at t and $g'_i(t) = f(t)$. Let $D = D_1 \cup D_2$, which is a countable set. Both g_1 and g_2 are continuous so $g = g_1 - g_2 : I \to X$ is continuous, and if $t \in I \setminus D$ then g is differentiable at t and $g'(t) = g'_1(t) - g'_2(t) = f(t) - f(t) = 0$. Then Corollary 11 shows that g is constant on I, i.e., that $g_1 - g_2$ is constant on I. □

We now give a construction of primitives of regulated functions.[4]

Theorem 13. *If $I = [a, b]$, X is a Banach space, and $f \in R(I, X)$, then the map $g : I \to X$ defined by $g(t) = \int_a^t f$ is a primitive of f on I.*

Proof. For $t \in [a, b)$ and $\epsilon > 0$, because f is regulated there is some $0 < \delta < b - t$ such that $0 < r \leq \delta$ implies that $\| f(t + r) - f(t^+) \| \leq \epsilon$. For $0 < r \leq \delta$ and for any $0 < \eta < r$, using Lemma 8 we have

$$\begin{aligned}
\left\| \int_a^{t+r} f - \int_a^t f - \int_t^{t+r} f(t^+) \right\| &= \left\| \int_t^{t+r} f - \int_t^{t+r} f(t^+) \right\| \\
&= \left\| \int_t^{t+\eta} (f - f(t^+)) + \int_{t+\eta}^{t+r} (f - f(t^+)) \right\| \\
&\leq \eta \sup_{t \leq s \leq t+\eta} \left\| f(s) - f(t^+) \right\| \\
&\quad + (r - \eta) \sup_{t+\eta \leq s \leq t+r} \left\| f(s) - f(t^+) \right\| \\
&\leq 2 \| f \|_\infty \eta + (r - \eta)\epsilon.
\end{aligned}$$

[4] Jean-Paul Penot, *Calculus Without Derivatives*, p. 124, Theorem 2.19.

This is true for all $0 < \eta < r$, so

$$\left\| \int_a^{t+r} f - \int_a^t f - \int_t^{t+r} f(t^+) \right\| \leq r\epsilon,$$

i.e.

$$\left\| \frac{g(t+r) - g(t)}{r} - f(t^+) \right\| \leq \epsilon.$$

This shows that

$$g'_+(t) = f(t^+).$$

Similarly,

$$g'_-(t) = f(t^-).$$

Because f is regulated, Lemma 4 shows that there is a countable set $D \subset I$ such that $t \in I \setminus D$ implies that f is continuous at t. Therefore, if $t \in I \setminus D$ then $f(t^+) = f(t^-) = f(t)$, so $g'_+(t) = g'_-(t)$, which means that if $t \in I \setminus D$ then g is differentiable at t, with $g'(t) = f(t)$. To prove that g is a primitive of f on I it suffices now to show that g is continuous. For $\epsilon > 0$ and $t \in I$, let $\delta = \frac{\epsilon}{\|f\|_\infty}$, and then for $|s - t| < \delta$ we have by Lemma 8 that

$$\|g(s) - g(t)\| = \left\| \int_a^s f - \int_a^t f \right\| = \left\| \int_s^t f \right\| \leq |t - s| \, \|f\|_\infty < \delta \, \|f\|_\infty = \epsilon,$$

showing that g is continuous at t, completing the proof. □

Suppose that X is a Banach space and that $f : [a, b] \to X$ is a primitive of a regulated function $h : [a, b] \to X$. Because h is regulated, by Theorem 13 the function $g : [a, b] \to X$ defined by $g(t) = \int_a^t f$ is a primitive of f on $[a, b]$. Then applying Lemma 12, there is some $c \in X$ such that $f(t) - g(t) = c$ for all $t \in [a, b]$. But $f(a) - g(a) = f(a)$, so $c = f(a)$. Hence, for all $t \in [a, b]$,

$$f(t) = f(a) + \int_a^t h.$$

But

$$\int_a^t h = \int_a^{a+\eta_1} h + \int_{a+\eta_1}^{t-\eta_2} h + \int_{t-\eta_2}^t h = \int_a^{a+\eta_1} h + \int_{a+\eta_1}^{t-\eta_2} f' + \int_{t-\eta_2}^t h$$

and

$$\left\| \int_a^{a+\eta_1} h \right\| \leq \eta_1 \|h\|_\infty, \qquad \left\| \int_{t-\eta_2}^t h \right\| \leq \eta_2 \|h\|_\infty,$$

hence as $\eta_1 \to 0^+$ and $\eta_2 \to 0^+$,

$$\int_{a+\eta_1}^{t-\eta_2} f' \to \int_a^t h,$$

and so it makes sense to write

$$\int_a^t f' = \int_a^t h,$$

and thus for all $t \in [a, b]$,

$$f(t) = f(a) + \int_a^t f'.$$