Unbounded operators, resolvents, the Friedrichs extension, and the Laplacian on $L^2(\mathbb{T}^d)$

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1 Unbounded operators

Let V be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, linear in the first argument. We write $|v|^2 = \langle v, v \rangle$ and for a bounded operator A on V we write

$$\|A\| = \sup_{|v| \leq 1} |Av|.$$

By an **operator** T, D_T **in** V we mean that D_T is a linear subspace of V and $T : D_T \to V$ is a linear map. For operators T, D_T and $T', D_{T'}$, by $T \subset T'$ we mean that $D_T \subset D_{T'}$ and the restriction of T' to D_T is equal to T, and we say that T' is an **extension** of T.

Lemma 1. *If X and Y are Banach spaces, X_0 is a dense linear subspace of X, and $T_0 : X_0 \to Y$ is a bounded operator, then there is a unique bounded operator $T : X \to Y$ whose restriction to X_0 is equal to T_0 and which satisfies $\|T\| = \|T_0\|$.*

Proof. For $x \in X$, let x_n, x'_n be sequences in X_0 tending to x, and thus $|x_n - x'_n| \leq |x_n - x| + |x'_n - x| \to 0$. $|T_0 x_n - T_0 x_m| \leq \|T_0\| \, |x_n - x_m|$, so $T_0 x_n$ is a Cauchy sequence in Y and hence converges to some $y \in Y$, and likewise $T_0 x'_n$ converges to some $y' \in Y$. Then

$$|y - y'| \leq |y - T_0 x_n| + \|T_0\| \, |x_n - x'_n| + |T_0 x'_n - y'| \to 0,$$

showing that $y = y'$. Therefore it makes sense to define $Tx = y$. Check that $T : X \to Y$ is linear. For $x \in X$, because $T_0 x_n \to Tx$ and $|x_n| \to |x|$,

$$|Tx| \leq |Tx - T_0 x_n| + |T_0 x_n| \leq |Tx - T_0 x_n| + \|T_0\| \, |x_n| \to \|T_0\| \, |x|,$$

showing that $\|T\| \leq \|T_0\|$, and so $T : X \to Y$ is a bounded operator.

For $x \in X_0$, $Tx = T_0 x$, which means that the restriction of T to X_0 is equal to T_0. Furthermore, $|T_0 x| = |Tx| \leq \|T\| \, |x|$, which shows that $\|T_0\| \leq \|T\|$, and so with $\|T\| \leq \|T_0\|$ we have $\|T\| = \|T_0\|$, completing the proof. □

For a densely defined operator T, D_T, let D_{T^*} be the set of those $w \in V$ such that $v \mapsto \langle Tv, w \rangle$ is continuous $D_T \to \mathbb{C}$. It is apparent that D_{T^*} is a linear subspace of V. For $w \in D_{T^*}$, by Lemma 1 there is some $\Lambda_w \in V^*$ whose restriction to D_T is equal to $v \mapsto \langle Tv, w \rangle$, and then by the Riesz representation theorem there is a unique $v_w \in V$ such that $\Lambda_w v = \langle v, v_w \rangle$ for all $v \in V$. Thus for $v \in D_T$ it holds that $\langle Tv, w \rangle = \langle v, v_w \rangle$, and if $u \in V$ also satisfies $\langle Tv, w \rangle = \langle v, u \rangle$ for all $v \in D_T$ then $\langle v, v_w \rangle = \langle v, u \rangle$ for all $v \in D_T$, which means that $v_w - u \in D_T^\perp$ and because D_T is dense it follows that $u = v_w$. Therefore it makes sense to define $T^* : D_{T^*} \to V$, called the **adjoint** of T, by

$$\langle Tv, w \rangle = \langle v, T^* w \rangle, \qquad v \in D_T, \quad w \in D_{T^*}.$$

$T^* : D_{T^*} \to V$ is a linear map. We shall only speak about the adjoint of a densely defined operator.

We call a densely defined operator T **self-adjoint** when $T = T^*$. An operator T, D_T is called **symmetric** when

$$\langle Tv, w \rangle = \langle v, Tw \rangle, \qquad v, w \in D_T,$$

and called **positive** if it is symmetric and satisfies

$$\langle Tv, v \rangle \geq 0, \qquad v D_T.$$

If T, D_T is symmetric then $\langle Tv, v \rangle \in \mathbb{R}$ for $v \in D_T$.

Lemma 2. *Let T, D_T be densely defined. T is symmetric if and only if $T \subset T^*$.*

Proof. If T is symmetric then for $w \in D_T$ the map $v \mapsto \langle Tv, w \rangle = \langle v, Tw \rangle$ is continuous $D_T \to \mathbb{C}$, hence $D_T \subset D_{T^*}$. Furthermore, for $w \in D_T \subset D_{T^*}$ it is the case that $\langle v, Tw \rangle = \langle Tv, w \rangle = \langle v, T^* w \rangle$ for all $v \in D_T$, hence $Tw - T^* w \in D_T^\perp$, and as D_T is dense this means that $Tw = T^* w$. Therefore, when T is densely defined and symmetric,

$$T \subset T^*.$$

On the other hand, if T is densely defined and $T \subset T^*$, then for $v, w \in D_T$, as $w \in D_{T^*}$ we have $\langle Tv, w \rangle = \langle v, T^* w \rangle = \langle v, Tw \rangle$, showing that T is symmetric. ☐

For an operator T, D_T, for $v, w \in D_T$ define the inner product $\langle \cdot, \cdot \rangle_T$ on the linear space D_T by

$$\langle v, w \rangle_T = \langle v, w \rangle + \langle Tv, Tw \rangle,$$

and write $|v|_T^2 = \langle v, v \rangle_T = |v|^2 + |Tv|^2$. An operator T, D_T is called **closed** if

$$\text{graph } T = \{(v, Tv) : v \in D_T\}$$

is a closed linear subspace of $V \times V$.

Lemma 3. *An operator T, D_T is closed if and only if D_T with the inner product $\langle \cdot, \cdot \rangle_T$ is a Hilbert space.*

Proof. Suppose that T is closed and let v_n be a Cauchy sequence in the norm $|\cdot|_T$. Then v_n and Tv_n are Cauchy sequences in the norm $|\cdot|$, and hence there are $v, w \in V$ such that $|v_n - v| \to 0$ and $|Tv_n - w| \to 0$. Thus $|(v_n, Tv_n) - (v, w)|^2 = |v_n - v|^2 + |Tv_n - w|^2 \to 0$, and because graph T is closed this means that $(v, w) \in \text{graph } T$, i.e. $v \in D_T$ and $w = Tv$. Therefore $|v_n - v|_T^2 = |v_n - v|^2 + |Tv_n - Tv|^2 = |v_n - v|^2 + |Tv_n - w|^2 \to 0$, showing that the Cauchy sequence v_n converges to v in the norm $|\cdot|_T$.

Suppose that $(D_T, \langle \cdot, \cdot \rangle_T)$ is a Hilbert space and let (v_n, Tv_n) be a sequence in graph T that converges to some $(v, w) \in V \times V$. This means that v_n converges to v in the norm $|\cdot|$ and Tv_n converges to w in the norm $|\cdot|$. Therefore v_n is a Cauchy sequence in the norm $|\cdot|_T$, and because $(D_T, \langle \cdot, \cdot \rangle_T)$ is a Hilbert space, there is some $u \in D_T$ to which v_n converges in the norm $|\cdot|_T$. That is, $v_n \to u$ in the norm $|\cdot|$ and $Tv_n \to Tu$ in the norm $|\cdot|$. But we already have that $v_n \to v$ and $Tv_n \to w$ in the norm $|\cdot|$, which implies that $u = v$ and $Tu = w$, so $v \in D_T$ and $w = Tv$, which means that $(v, w) \in \text{graph } T$. □

Define $J : V \times V \to V \times V$ by

$$J(v, w) = (-w, v).$$

$J^2 = I$, and J is a unitary operator:

$$\begin{aligned} \langle J(v_1, v_2), J(w_1, w_2) \rangle &= \langle (-v_2, v_1), (-w_2, w_1) \rangle \\ &= \langle -v_2, -w_2 \rangle + \langle v_1, w_1 \rangle \\ &= \langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle \\ &= \langle (v_1, v_2), (w_1, w_2) \rangle . \end{aligned}$$

Lemma 4. *If T is a densely defined operator then*

$$\text{graph } T^* = (J\text{graph } T)^\perp.$$

Proof. $(x, y) \in \text{graph } T^*$ is equivalent to $x \in D_{T^*}$ and $y = T^*x$ is equivalent to $\langle Tv, x \rangle = \langle v, y \rangle$ for all $v \in D_T$ is equivalent to $\langle (-Tv, v), (x, y) \rangle = 0$ for all $v \in D_T$ is equivalent to $\langle J(v, w), (x, y) \rangle = 0$ for all $(v, w) \in \text{graph } T$ is equivalent to $(x, y) \in (J(\text{graph } T))^\perp$. □

The above lemma shows that an adjoint T^* is a closed operator.

Lemma 5. *If T is a closed densely defined operator, then*

$$V \times V = (J\text{graph } T) \oplus \text{graph } T^*$$

is an orthogonal direct sum.

Proof. Generally, if M is a linear subspace of $V \times V$,

$$V \times V = \overline{M} \oplus M^\perp$$

is an orthogonal direct sum. For $M = J\text{graph } T$, because graph T is a closed linear subspace of $V \times V$, so is M. Thus

$$V \times V = (J\text{graph } T) \oplus (J\text{graph } T)^{\perp}.$$

By Lemma 4 this is

$$V \times V = (J\text{graph } T) \oplus \text{graph } T^*.$$

□

An operator T, D_T is called **closable** if there exists a closed extension of it. If T, D_T is closable with closed extensions T_1, D_{T_1} and T_2, D_{T_2} and $(v, w) \in \overline{\text{graph } T}$, then $(v, w) \in \text{graph } T_1 \cap \text{graph } T_2$, so $T_1 v = w$ and $T_2 v = w$, showing that the restriction of T_1 to D_T is equal to the restriction of T_2 to D_T. Therefore it makes sense to define $\overline{T}$ to be the intersection of all closed extensions of T. We call $\overline{T}, D_{\overline{T}}$ the **closure of** T. For $\pi_1(v, w) = v$, $D_{\overline{T}} = \pi_1(\overline{\text{graph } T})$, and $\overline{T}$ is the restriction of any closed extension of T to $D_{\overline{T}}$. In other words, if an operator is closable then there exists a minimal closed extension of it, called its closure and denoted $\overline{T}$. For a densely defined symmetric operator T it holds that $T \subset T^*$. But T^* is closed, showing that a densely defined symmetric operator is closable.

Lemma 6. *An operator T, D_T is closable if and only if whenever v_n is a sequence in D_T with $v_n \to 0$ and $T v_n \to v$, it follows that $v = 0$.*

Proof. Suppose that T is closable, with a closed extension $T', D_{T'}$. As $v_n \to 0$ and $T' v_n = T v_n \to v$ it holds that $(v_n, T' v_n) \to (0, v)$, and because T' is closed, $v = T' 0 = 0$.

Now let D be the set of those $v \in V$ for which there is a sequence $v_n \in D_T$ such that $v_n \to v$ and $T v_n$ converges to something in V. This is a linear subspace of V. If $v_n \to v, T v_n \to x$ and $w_n \to v, T w_n \to y$, then $v_n - w_n \to 0$ and $T(v_n - w_n) = T v_n - T w_n \to x - y$, so by hypothesis $x - y = 0$, i.e. $x = y$. Therefore it makes sense to define $T' : D \to V$ by: for $v \in D$ there is a sequence $v_n \in D_T$ that tends to v, and $T' v$ is the limit of $T v_n$. Check that T' is linear. If $(v_n, T' v_n) \in \text{graph } T'$ tends to $(x, y) \in V \times V$, then for each n there is some $w_n \in D_T$ with $|w_n - v_n| + |T w_n - T' v_n| \leq \frac{1}{n}$. Then $|x - w_n| \leq |x - v_n| + |w_n - v_n| \to 0$ and $|y - T w_n| \leq |y - T' v_n| + |T w_n - T' v_n| \to 0$, meaning that $w_n \to x$ and $T w_n \to y$. By the definition of D this means that $x \in D$. Moreover, $T' x$ is the limit of $T w_n$, i.e. $T' x = y$, showing that $(x, y) \in \text{graph } T'$. Hence T' is a closed operator. It is immediate that $D_T \subset D$ and that for $v \in D_T$, $T' v$ is the limit of the constant sequence $T v$, namely $T' v = T v$, showing that $T \subset T'$. Therefore T', D is a closed extension of T, D_T. □

Lemma 7. *If S and T are densely defined operators with $S \subset T$, then $T^* \subset S^*$.*

If T is a densely defined closable operator then $\overline{T}^* = T^*$.

Proof. $S \subset T$ implies $J\text{graph } S \subset J\text{graph } T$ implies $(J\text{graph } T)^\perp \subset (J\text{graph } S)^\perp$ implies by Lemma 4

$$\text{graph } T^* \subset \text{graph } S^*.$$

If T is densely defined and closable, then $T \subset \overline{T}$ so by the above $\overline{T}^* \subset T^*$. We now prove that $T^* \subset \overline{T}^*$. Take $w \in D_{T^*}$. For all $v \in D_T$ it holds that $\langle Tv, w\rangle = \langle v, T^* w\rangle$. For $x \in D_{\overline{T}}$, because $(x, \overline{T}x) \in \text{graph } \overline{T} = \overline{\text{graph } T}$, there is a sequence $v_n \in D_T$ such that $(v_n, Tv_n) \to (x, \overline{T}x)$. Since $Tv_n \to \overline{T}x$ and $v_n \to x$,

$$\left\langle \overline{T}x, w\right\rangle = \lim_{n\to\infty} \langle Tv_n, w\rangle = \lim_{n\to\infty} \langle v_n, T^* w\rangle = \langle x, T^* w\rangle\,,$$

which shows that $x \mapsto \left\langle \overline{T}x, w\right\rangle = \overline{x}T^* w$ is continuous $D_{\overline{T}} \to \mathbb{C}$. This means that $w \in D_{\overline{T}^*}$. Moreover, $\left\langle \overline{T}x, w\right\rangle = \left\langle x, \overline{T}^* w\right\rangle$ for all $x \in D_{\overline{T}}$ and $\left\langle \overline{T}x, w\right\rangle = \langle x, T^* w\rangle$ for all $x \in D_{\overline{T}}$, hence $\left\langle x, \overline{T}^* w - T^* w\right\rangle = 0$ for all $x \in D_{\overline{T}}$, and because $D_{\overline{T}}$ is dense this implies that $\overline{T}^* w = T^* w$. Therefore $T^* \subset \overline{T}^*$. □

Lemma 8. *Let T, D_T be a densely defined operator. T is closable if and only if T^* is densely defined, and in this case $\overline{T} = T^{**}$.*

Proof. Suppose that T^* is densely defined, and then T^{**} makes sense and is a closed operator. For $v \in D_T$ and $w \in D_{T^*}$,

$$\langle T^* w, v\rangle = \overline{\langle v, T^* w\rangle} = \overline{\langle Tv, w\rangle} = \langle w, Tv\rangle\,,$$

which shows that $w \mapsto \langle T^* w, v\rangle$ is continuous $D_{T^*} \to \mathbb{C}$ and hence that $v \in D_{T^{**}}$. Furthermore, $\langle w, T^{**}v\rangle = \langle T^* w, v\rangle = \langle w, Tv\rangle$ and so $\langle w, T^{**}v - Tv\rangle$ for all $w \in D_{T^*}$, and because D_{T^*} is dense in V this implies that $T^{**}v = Tv$. Therefore $T \subset T^{**}$, and T^{**} is closed so T is closable.

Suppose that T is closable and let $w \in D_{T^*}^\perp$; showing that $w = 0$ will prove that T^* is densely defined. For $(x, y) \in \text{graph } T^*$ it then holds that $\langle (x, y), (w, 0)\rangle = \langle x, w\rangle + \langle y, 0\rangle = \langle x, w\rangle = 0$, meaning $(w, 0) \in (\text{graph } T^*)^\perp$. Applying Lemma 4,

$$(w, 0) \in (J\text{graph } T)^{\perp\perp} = \overline{J\text{graph } T} = J\overline{\text{graph } T}.$$

Because T is closable, $\overline{\text{graph } T} = \text{graph } \overline{T}$ is a linear space, so $-\text{graph } \overline{T} = \text{graph } \overline{T}$. Then

$$(0, w) = (-0, w) = J(w, 0) \in J^2\text{graph } \overline{T} = -\text{graph } \overline{T} = \text{graph } \overline{T}.$$

$(0, w) \in \text{graph } \overline{T}$ means that $\overline{T}0 = w$, so $w = 0$. Therefore T^* is densely defined.

If T is densely defined and closable, then because T^* is densely defined, Lemma 4 says $\text{graph } T^{**} = (J\text{graph } T^*)^\perp$. But also by applying Lemma 4, $(J\text{graph } T^*)^\perp = (J(J\text{graph } T)^\perp)^\perp$; check that $(JM)^\perp = JM^\perp$ for M a linear subspace of $V \times V$, and thus

$$\text{graph } T^{**} = (J^2(\text{graph } T)^\perp)^\perp = (-(\text{graph } T)^\perp)^\perp = (\text{graph } T)^{\perp\perp} = \overline{\text{graph } T}.$$

Because T is closable this means that $T^{**} = \overline{T}$. □

2 Resolvents

For an operator T, D_T in V and for $\lambda \in \mathbb{C}$, we write

$$T_\lambda = T - \lambda, \qquad \mathscr{R}_\lambda = T_\lambda D_T.$$

We define the **resolvent set** $\rho(T)$ of T as the set of those $\lambda \in \mathbb{C}$ such that (i) $T_\lambda : D_T \to$ is injective, (ii) $\mathscr{R}_\lambda$ is dense in V, and (iii) $T_\lambda^{-1} : \mathscr{R}_\lambda \to D_T$ is a bounded operator. For $\lambda \in \rho(T)$, because $\mathscr{R}_\lambda$ is a dense linear subspace of V and $T_\lambda^{-1} : \mathscr{R}_\lambda \to V$ is bounded, by Lemma 1 there is a unique bounded operator $R_\lambda : V \to V$ whose restriction to $\mathscr{R}_\lambda$ is equal to T_λ^{-1}, and $\|R_\lambda\| = \left\|T_\lambda^{-1}\right\|$. We call R_λ a **resolvent** of T.

We will use the following theorem to prove that the resolvent set is open.[1]

Theorem 9. *Let T, D_T be an operator in V and let $\lambda \in \mathbb{C}$. If $T_\lambda : D_T \to V$ is injective and $T_\lambda^{-1} : \mathscr{R}_\lambda \to D_T$ is bounded, then $\left\|T_\lambda^{-1}\right\| |\mu - \lambda| < 1$ implies that $T_\mu : D_T \to V$ is injective and $T_\mu^{-1} : \mathscr{R}_\mu \to D_T$ is bounded, and $\overline{\mathscr{R}_\mu}$ is not a proper subset of $\overline{\mathscr{R}_\lambda}$.*

Proof. For $x \in D_T$,

$$T_\mu x = Tx - \mu x = T_\lambda x + \lambda x - \mu x = T_\lambda x + (\lambda - \mu)x.$$

Hence $|T_\mu x| \geq |T_\lambda x| - |\lambda - \mu||x|$. But

$$|x| = |T_\lambda^{-1} T_\lambda x| \leq \left\|T_\lambda^{-1}\right\| |T_\lambda x|,$$

so

$$\left\|T_\lambda^{-1}\right\| |T_\mu x| \geq \left\|T_\lambda^{-1}\right\| |T_\lambda x| - \left\|T_\lambda^{-1}\right\| |\lambda - \mu||x| \geq |x| - \left\|T_\lambda^{-1}\right\| |\lambda - \mu||x|,$$

i.e.

$$\left\|T_\lambda^{-1}\right\| |T_\mu x| \geq |x|(1 - \left\|T_\lambda^{-1}\right\| |\lambda - \mu|). \tag{1}$$

Therefore, if $T_\mu x = 0$ then $x = 0$, showing that $T_\mu : D_T \to V$ is injective. For $y = T_\mu x \in \mathscr{R}_\mu$, applying (1) with $x = T_\mu^{-1} y$,

$$|T_\mu^{-1} y| \leq (1 - \left\|T_\lambda^{-1}\right\| |\lambda - \mu|)^{-1} \left\|T_\lambda^{-1}\right\| |T_\mu T_\mu^{-1} y| = (1 - \left\|T_\lambda^{-1}\right\| |\lambda - \mu|)^{-1} \left\|T_\lambda^{-1}\right\| |y|,$$

showing that $T_\mu^{-1} : \mathscr{R}_\mu \to D_T$ is bounded.

Riesz's lemma states that if X is a normed space and X_0 is a closed linear subspace of X with $X_0 \neq X$, then for each $0 < \theta < 1$ there is some $x_\theta \in X$ with $|x_\theta| = 1$ and $|x - x_\theta| \geq \theta$ for all $x \in X_0$.[2] Assume by contradiction that $\overline{\mathscr{R}_\mu}$ is a proper subset of $\overline{\mathscr{R}_\lambda}$. Take

$$\left\|T_\lambda^{-1}\right\| |\mu - \lambda| < \theta < 1,$$

[1] Angus E. Taylor, *Introduction to Functional Analysis*, p. 256, Theorem 5.1-A.
[2] Angus E. Taylor, *Introduction to Functional Analysis*, p. 96, Theorem 3.12-E.

and applying Riesz's lemma there is some $y_\theta \in \overline{\mathscr{R}_\lambda}$ such that $|y_\theta| = 1$ and such that $|y - y_\theta| \geq \theta$ for all $y \in \overline{\mathscr{R}_\mu}$. Take $x_n \in D_T$ with $T_\lambda x_n \to y_\theta$. As $T_\mu x_n = T_\lambda x_n + (\lambda - \mu)x_n$, we have

$$|T_\mu x_n - T_\lambda x_n| = |\lambda - \mu||T_\lambda^{-1} T_\lambda x_n| \leq |\lambda - \mu| \left\|T_\lambda^{-1}\right\| |T_\lambda x_n|.$$

Now, $T_\mu x_n \in \mathscr{R}_\mu$ so $|T_\mu x_n - y_\theta| \geq \theta|$, and hence

$$\theta \leq |T_\mu x_n - T_\lambda x_n| + |T_\lambda x_n - y_\theta| \leq |\lambda - \mu| \left\|T_\lambda^{-1}\right\| |T_\lambda x_n| + |T_\lambda x_n - y_\theta|.$$

As $n \to \infty$, $T_\lambda x_n \to y_\theta$, so the above right-hand side tends to $|\lambda - \mu| \left\|T_\lambda^{-1}\right\| |y_\theta|$. Hence

$$\theta \leq |\lambda - \mu| \left\|T_\lambda^{-1}\right\| |y_\theta| = |\lambda - \mu| \left\|T_\lambda^{-1}\right\|,$$

a contradiction. Therefore $\overline{\mathscr{R}_\mu}$ is not a proper subset of $\overline{\mathscr{R}_\lambda}$. □

Corollary 10. *For an operator T, D_T in V, if $\lambda \in \rho(T)$ then $\left\|T_\lambda^{-1}\right\| |\mu - \lambda| < 1$ implies that $\mu \in \rho(T)$. In particular, $\rho(T)$ is an open subset of $\mathbb{C}$.*

Proof. If $\lambda \in \rho(T)$, then $T_\lambda : D_T \to V$ is injective and $T_\lambda^{-1} : \mathscr{R}_\lambda \to D_T$ is bounded, so by Theorem 9, $|\mu - \lambda| < \left\|T_\lambda^{-1}\right\|^{-1}$ implies that $T_\mu : D_T \to V$ is injective, $T_\mu^{-1} : \mathscr{R}_\mu \to D_T$ is bounded, and $\overline{\mathscr{R}_\mu}$ is not a proper subset of $\overline{\mathscr{R}_\lambda}$. But because $\lambda \in \rho(T)$ it is the case that $\overline{\mathscr{R}_\lambda} = V$, so $\overline{\mathscr{R}_\mu}$ is not a proper subset of V, i.e. $\overline{\mathscr{R}_\mu} = V$. This shows that $\mu \in \rho(T)$. □

We characterize the resolvent sets of closed operators in the following lemma.[3]

Lemma 11. *Let T, D_T be a closed operator. For $\lambda \in \mathbb{C}$, the following are equivalent:*

1. *$\lambda \in \rho(T)$.*
2. *$T - \lambda : D_T \to V$ is a bijection.*
3. *There is a bounded operator R on V such that*

$$R(T - \lambda) = I_{D_T}, \qquad (T - \lambda)R = I_V.$$

Proof. Suppose $\lambda \in \rho(T)$ and take $x \in V$. Because $(T - \lambda)D_T$ is dense in V there is a sequence y_n in D_T such that $(T - \lambda)y_n \to x$. Furthermore, $R_\lambda : V \to V$ is continuous, so $R_\lambda(T - \lambda)y_n \to R_\lambda x$. But $R_\lambda(T - \lambda)y_n = y_n$, so $y_n \to R_\lambda x$. $y_n \to R_\lambda x$ and $(T - \lambda)y_n \to x$ yield $Ty_n \to x + \lambda R_\lambda x$, and thence $(y_n, Ty_n) \to (R_\lambda x, x + \lambda R_\lambda x)$. But $y_n \in D_T$ and graph T is closed, which means that $R_\lambda x \in D_T$ and $TR_\lambda x = x + \lambda R_\lambda x$. That is, $R_\lambda x \in D_T$ and $(T - \lambda)R_\lambda x = x$, which implies that $x = (T - \lambda)R_\lambda x \in (T - \lambda)D_T$, showing that $T - \lambda$ is surjective. We already know that $T - \lambda$ is injective, so we have proved that $T - \lambda : D_T \to V$ is a bijection.

[3] Gilles Royer, *An Initiation to Logarithmic Sobolev Inequalities*, p. 2, Proposition 1.1.4.

Suppose that $T-\lambda : D_T \to V$ is a bijection. Because T is closed, Lemma 3 states that the linear space D_T with the inner product $\langle v, w\rangle_T = \langle v, w\rangle + \langle Tv, Tw\rangle$ is a Hilbert space. But $|Tv|^2 = \langle Tv, Tv\rangle \leq \langle v, v\rangle_T = |v|^2$, so T is bounded $(D_T, \langle\cdot,\cdot\rangle_T) \to V$, and $|\lambda v|^2 = \langle \lambda v, \lambda v\rangle = |\lambda|^2 \langle v, v\rangle \leq |\lambda|^2 |v|_T^2$, so $v \mapsto \lambda v$ is bounded $(D_T, \langle\cdot,\cdot\rangle_T) \to V$. Because $T-\lambda$ is a bijective bounded operator $(D_T, \langle\cdot,\cdot\rangle_T) \to V$, the open mapping theorem tells us that $(T-\lambda)^{-1} : V \to (D_T, \langle\cdot,\cdot\rangle_T)$ is bounded. Because $|\cdot| \leq |\cdot|_T$, a fortiori $(T-\lambda)^{-1} : V \to (D_T, \langle\cdot,\cdot\rangle)$ is bounded.

Suppose that there is a bounded operator R in V such that

$$R(T-\lambda) = I_{D_T}, \qquad (T-\lambda)R = I_V.$$

The first equality implies that $T-\lambda : D_T \to V$ is injective. The second equality implies that $T-\lambda : D_T \to V$ is surjective, and a fortiori that $(T-\lambda)D_T$ is dense in V. For $w \in (T-\lambda)D_T$, as $(T-\lambda)^{-1}w = Rw$ and as R is a bounded operator, $|(T-\lambda)^{-1}w| = |Rw| \leq \|R\|\,|w|$, showing that $(T-\lambda)^{-1} : (T-\lambda)D_T \to V$ is a bounded operator. This establishes that $\lambda \in \rho(T)$. □

The hypothesis of the following theorem is satisfied if T, D_T is a closed operator.[4] We denote by $\mathscr{L}(V)$ the complex Banach algebra of bounded linear operators $V \to V$

Theorem 12 (Resolvent identity)**.** *Suppose that T, D_T is an operator in V such that $\mathscr{R}_\lambda = V$ for each $\lambda \in \rho(T)$. If $\lambda, \mu \in \rho(T)$, then*

$$R_\lambda - R_\mu = (\lambda-\mu)R_\lambda R_\mu.$$

For $\lambda, \mu \in \rho(T)$ and $n \geq 0$,

$$R_\lambda = \sum_{k=0}^{n} (\lambda-\mu)^k R_\mu^{k+1} + (\lambda-\mu)^{n+1} R_\mu^{n+1} R_\lambda. \tag{2}$$

If $|\lambda-\mu|\,\|R_\mu\| < 1$, then

$$\sum_{k=0}^{n} (\lambda-\mu)^k R_\mu^{k+1} \to R_\lambda$$

in the operator norm. The function $\lambda \mapsto R_\lambda$ is holomorphic $\rho(T) \to \mathscr{L}(V)$, and

$$\frac{d}{d\lambda} R_\lambda = R_\lambda^2.$$

Proof. For $y \in V$, by hypothesis there is some $x \in D_T$ with $y = T_\mu x$, $x = R_\mu y$. Because $T_\mu x - T_\lambda x = (\lambda-\mu)x$,

$$y - T_\lambda R_\mu y = (\lambda-\mu)R_\mu y.$$

[4] Angus E. Taylor, *Introduction to Functional Analysis*, p. 257, Theorem 5.1-C.

Applying R_λ on the left,

$$R_\lambda y - R_\lambda T_\lambda R_\mu y = R_\lambda(\lambda - \mu)R_\mu y,$$

i.e.

$$R_\lambda y - R_\mu y = (\lambda - \mu)R_\lambda R_\mu y.$$

This shows that $R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu$.

The resolvent identity provides $R_\lambda = R_\mu + (\lambda - \mu)R_\mu R_\lambda$. Assume by induction that for some n, (2) is true. Then, using the resolvent identity $R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu$ and $R_\lambda R_\mu = R_\mu R_\lambda$ (which is immediate from the resolvent identity),

$$\begin{aligned}
R_\lambda &= \sum_{k=0}^{n+1}(\lambda-\mu)^k R_\mu^{k+1} - (\lambda-\mu)^{n+1}R_\mu^{n+2} + (\lambda-\mu)^{n+1}R_\mu^{n+1}R_\lambda \\
&= \sum_{k=0}^{n+1}(\lambda-\mu)^k R_\mu^{k+1} + (\lambda-\mu)^{n+1}R_\mu^{n+1}(-R_\mu + R_\lambda) \\
&= \sum_{k=0}^{n+1}(\lambda-\mu)^k R_\mu^{k+1} + (\lambda-\mu)^{n+1}R_\mu^{n+1}\cdot(\lambda-\mu)R_\lambda R_\mu \\
&= \sum_{k=0}^{n+1}(\lambda-\mu)^k R_\mu^{k+1} + (\lambda-\mu)^{n+2}R_\mu^{n+2}R_\lambda,
\end{aligned}$$

showing that (2) is true for $n+1$.

If $r = |\mu - \lambda| \, \|R_\lambda\| < 1$, then

$$\left\|(\lambda-\mu)^{n+2}R_\mu^{n+2}R_\lambda\right\| \le |\lambda-\mu|^{n+2}\,\|R_\mu\|^{n+2}\,\|R_\lambda\| = \|R_\lambda\|\,r^{n+2},$$

which tends to 0 as $n \to \infty$, and thus (2) implies $\sum_{k=0}^{n}(\lambda-\mu)^k R_\mu^{k+1} \to R_\lambda$ in $\mathscr{L}(V)$.

Take $\lambda \in \rho(T)$. For $\mu \in \rho(T)$ with $\mu \neq \lambda$, applying the resolvent identity twice yields

$$\frac{R_\mu - R_\lambda}{\mu - \lambda} - R_\lambda^2 = R_\mu R_\lambda - R_\lambda^2 = (R_\mu - R_\lambda)R_\lambda = (\mu - \lambda)R_\mu R_\lambda R_\lambda.$$

Suppose that μ satisfies $\|R_\lambda\|\,|\mu - \lambda| \le \frac{1}{2}$. Then $\mu \in \rho(T)$ by Corollary 10. From the resolvent identity, $\|R_\lambda - R_\mu\| \le |\lambda - \mu|\,\|R_\lambda\|\,\|R_\mu\|$, and using this with $\|R_\mu - R_\lambda\| \ge \|R_\mu\| - \|R_\lambda\|$ gives

$$\|R_\mu\|\,(1 - |\lambda-\mu|\,\|R_\lambda\|) \le \|R_\lambda\|\,. \tag{3}$$

Because $\left\|T_\lambda^{-1}\right\|\,|\mu - \lambda| \le \frac{1}{2}$,

$$\|R_\mu\| \le 2\,\|R_\lambda\|\,,$$

and using this with (3) yields

$$\|R_\lambda - R_\mu\| \leq 2|\lambda - \mu| \, \|R_\lambda\|^2 .$$

This shows that $\mu \mapsto R_\mu$ is a continuous function from the closed disc with radius $\frac{1}{2}\|R_\lambda\|^{-1}$ and center λ to $\mathscr{L}(V)$. Let $\|R_\mu\| \leq M$ for all μ in this compact disc. Hence

$$\left\| \frac{R_\mu - R_\lambda}{\mu - \lambda} - R_\lambda^2 \right\| \leq |\mu - \lambda| \, \|R_\mu\| \, \|R_\lambda\|^2 \leq M \|R_\lambda\|^2 |\mu - \lambda|,$$

which tends to 0 as $\mu \to \lambda$. Therefore $\frac{R_\mu - R_\lambda}{\mu - \lambda}$ tends to R_λ^2 in $\mathscr{L}(V)$ as $\mu \to \lambda$, showing that R_λ is holomorphic $\rho(T) \to \mathscr{L}(V)$. □

Lemma 13. *If T, D_T is a self-adjoint operator in V and $\lambda = x + iy$, then for $y \neq 0$ it is the case that $\lambda \in \rho(T)$ and $\|R_\lambda\| \leq 1/|y|$. If furthermore T is positive, then for $y = 0$ and $x < 0$, it is the case that $\lambda \in \rho(T)$ and $\|R_\lambda\| \leq 1/|x|$.*

Proof. Write $\lambda = x + iy$. If $y \neq 0$, then for $v \in D_T$, using that $T - x$ is symmetric,

$$\begin{aligned} |(T - \lambda)v|^2 &= \langle (T - x)v - iyv, (T - x)v - iyv \rangle \\ &= |(T - x)v|^2 + iy \langle (T - x)v, v \rangle - iy \langle v, (T - x)v \rangle + y^2 |v|^2 \\ &= |(T - x)v|^2 + y^2 |v|^2 \\ &\geq y^2 |v|^2. \end{aligned}$$

Since $y \neq 0$, if $v \neq 0$ then $(T - \lambda)v \neq 0$, showing that $T - \lambda$ is injective. If $w \in ((T - \lambda)D_T)^\perp$, then for $v \in D_T$ it holds that $\langle (T - \lambda)v, w \rangle = 0 = \langle v, 0 \rangle$, so $v \mapsto \langle (T - \lambda)v, w \rangle$ is continuous $D_T \to \mathbb{C}$, meaning that $w \in D_{T^*} = D_T$. Furthermore, $(T - \lambda)^* w = 0$, meaning $Tw = T^* w = \overline{\lambda} w$. Then $\langle Tw, w \rangle = \langle \overline{\lambda} w, w \rangle = \overline{\lambda} \langle w, w \rangle$, and because T is symmetric $\langle Tw, w \rangle \in \mathbb{R}$, implying that $w = 0$. This establishes that $(T - \lambda)D_T$ is dense in V. Define $F : \text{graph}\,(T - \lambda) \to (T - \lambda)D_T$ by $F(v, w) = w$. For $v \in D_T$,

$$|(T - \lambda)v|^2 \leq |(T - \lambda)v|^2 + |v|^2 \leq (1 + y^{-2}) |(T - \lambda)v|^2,$$

and because F is surjective this implies that $F : \text{graph}\,(T - \lambda) \to (T - \lambda)D_T$ and $F^{-1} : (T - \lambda)D_T \to \text{graph}\,(T - \lambda)$ are Lipschitz, meaning that graph $(T - \lambda)$ and $(T - \lambda)D_T$ are **Lipschitz equivalent**. Because T is self-adjoint it is closed, and then $T - \lambda$ is a closed operator, because λI is a bounded operator, and therefore graph $(T - \lambda)$ is a complete metric, being a closed set in the complete metric space $V \times V$. Since graph $(T - \lambda)$ and $(T - \lambda)D_T$ are Lipschitz equivalent, it follows that $(T - \lambda)D_T$ is a complete metric space. Because $(T - \lambda)D_T$ is a complete subspace of the metric space V, it is a closed set in V. But we have proved that $(T - \lambda)D_T$ is dense in V, so $(T - \lambda)D_T = V$, meaning that $T - \lambda : D_T \to V$ is surjective.

For $w \in V$, let v be the unique element of D_T for which $(T - \lambda)v = w$. $|(T - \lambda)v|^2 \geq y^2 |v|^2$ means that $|v| \leq |y|^{-1} |w|$, i.e. $|(T - \lambda)^{-1} w| \leq |y|^{-1} |w|$. This shows that $\lambda \in \rho(T)$ and that $\|R_\lambda\| \leq 1/|\mathrm{Im}\,\lambda|$. □

3 The Friedrichs extension

If T, D_T is a positive densely defined operator, for $v, w \in D_T$ define

$$(v, w)_T = \langle v, w \rangle + \langle Tv, w \rangle,$$

and write $(v)_T^2 = (v, v)_T$. As T is symmetric, $\overline{(w, v)_T} = \overline{\langle w, v \rangle + \langle w, Tv \rangle} = \langle v, w \rangle + \langle Tv, w \rangle = (v, w)_T$. As T is positive, $(v, v)_T = \langle v, v \rangle + \langle Tv, v \rangle \geq 0$. Therefore $(\cdot, \cdot)_T$ is an inner product on D_T.

Let V_T be the completion of D_T with respect to the inner product $(\cdot, \cdot)_T$. For $f \in V_T$, if $v_n, w_n \in D_T$ are Cauchy sequences that each tend to f in the norm $(\cdot)_T$ then on the one hand, v_n and w_n are Cauchy sequences in the norm $|\cdot|$ and hence converge in $|\cdot|$ respectively to some $v, w \in V$. On the other hand, $v_n - w_m$ converges to 0 in the norm $(\cdot)_T$ so $|v - w| \leq |v - v_n| + |v_n - w_n| + |w_n - w| \to 0$, showing that $v = w$. Thus for $f \in V_T$, which is the $(\cdot)_T$ limit of some Cauchy sequence $v_n \in D_T$, it makes sense to define $i_T f$ to be the $|\cdot|$ limit of v_n in V. Check that

$$i_T : V_T \to V$$

is a linear map. For $f \in V_T$, where $v_n \in D_T$ tends to f in the norm $(\cdot)_T$,

$$|i_T f| = |i_T f - v_n| + |v_n| \leq |i_T f - v_n| + (v_n)_T \to (f)_T,$$

which means that $\|i_T\| \leq 1$. If $i_T f = 0$, where $v_n \in D_T$ tends to f in the norm $(\cdot)_T$, then v_n tends to 0 in the norm $|\cdot|$, and for each $v \in D_T$ we have

$$(v, f)_T = \lim_{n\to\infty} (v, v_n)_T = \lim_{n\to\infty} (\langle v, v_n \rangle + \langle Tv, v_n \rangle) = 0.$$

Because D_T is dense in V_T, this implies that $f = 0$, showing that i_T is an injection.[5]

For $v \in V$ define $\lambda_v : V_T \to \mathbb{C}$ by $\lambda_v f = \langle i_T f, v \rangle$. This satisfies $|\lambda_v f| \leq |i_T f||v| \leq (f)_T |v|$, so $\|\lambda_v\| \leq |v|$. By the Riesz representation theorem there is a unique $Bv \in V_T$ satisfying

$$\langle i_T f, v \rangle = \lambda_v f = (f, Bv)_T$$

for $f \in V_T$, and $(Bv)_T = \|\lambda_v\| \leq |v|$. Check that the map

$$B : V \to V_T$$

is linear, and has operator norm $\|B\| \leq 1$. For $v, w \in V$,

$$\langle i_T Bv, w \rangle = \lambda_w Bv = (Bv, Bw)_T = \overline{(Bw, Bv)_T} = \overline{\lambda_v Bw} = \langle v, i_T Bw \rangle,$$

showing that $i_T B : V \to V$ is symmetric. For $v \in V$,

$$\langle i_T Bv, v \rangle = \lambda_v Bv = (Bv, Bv)_T \geq 0,$$

[5] Peter D. Lax, *Functional Analysis*, p. 403, §33.3.

showing that $i_T B$ is positive. If $Bv = 0$ then for $f \in V_T$,

$$\langle i_T f, v \rangle = \lambda_v f = (f, Bv)_T = (f, 0)_T = 0,$$

and because $D_T \subset i_T V_T$ and D_T is dense in V this implies that $v = 0$. This shows that $B : V \to V_T$ is injective, and because $i_T : V_T \to V$ is injective we get that $i_T B : V \to V$ is injective. If $w \in (i_T BV)^\perp$ then

$$0 = \langle i_T Bw, w \rangle = \lambda_w Bw = (Bw, Bw)_T,$$

which implies that $Bw = 0$, and because B is injective this implies $w = 0$. Therefore $(i_T BV)^\perp = \{0\}$, which means that $i_T BV$ is dense in V. We have so far established that $i_T B : V \to V$ has $\|i_T B\| \leq 1$, is positive, is an injection, and has dense image. Furthermore, $i_T B$ is bounded and symmetric it is self-adjoint.

Let $D_A = i_T BV$ and define $A : D_A \to V$ by

$$Ax = (i_T B)^{-1} x, \qquad x \in D_A = i_T BV,$$

which is a linear isomorphism $D_A \to V$. For $x = i_T Bv, y = i_T Bw$, because $i_T B$ is symmetric we get $\langle Ax, y \rangle = \langle v, i_T Bw \rangle = \langle i_T Bv, w \rangle = \langle x, Ay \rangle$, which means that A is symmetric. For $x = i_T Bv$,

$$\langle Ax, x \rangle = \langle x, Ax \rangle = \langle i_T Bv, v \rangle = (Bv, Bv)_T,$$

which shows a fortiori that A is positive. Define $S : V \times V \to V \times V$ by $S(v, w) = (w, v)$, and $A = (i_T B)^{-1}$ means

$$\text{graph } A = S(\text{graph } i_T B).$$

$J \circ S = -S \circ J$:

$$J(S(v, w)) = (-v, w), \qquad S(J(v, w)) = S(-w, v) = (v, -w).$$

A is densely defined, since $i_T B$ has dense image, so it makes sense to talk about the adjoint of A. Then

$$\begin{aligned}
\text{graph } A^* &= (J(\text{graph } A))^\perp \\
&= (JS(\text{graph } i_T B))^\perp \\
&= (-SJ(\text{graph } i_T B))^\perp \\
&= (SJ(\text{graph } i_T B))^\perp \\
&= S((J(\text{graph } i_T B))^\perp) \\
&= S(\text{graph } (i_T B)^*) \\
&= S(\text{graph } i_T B) \\
&= \text{graph } A,
\end{aligned}$$

showing that A is self-adjoint.

Now define $S = 1 + T$, which is symmetric because T is. For $v, w \in D_T$, as $i_T v = v$,

$$\langle v, Sw\rangle = \langle i_T v, Sw\rangle = (v, BSw)_T,$$

and

$$\langle v, Sw\rangle = \langle Sv, w\rangle = \langle v, w\rangle + \langle Tv, w\rangle = (v, w)_T,$$

and therefore $(v, w - BSw)_T = 0$ for $v \in D_T$. Because D_T is $(\cdot)_T$ dense in V_T this implies $BSw = w$ for $w \in D_T$. This shows that D_T is contained in the image of B and as $i_T D_T = D_T$, implies that $D_S = D_T \subset D_A$. For $w \in D_S$, as $i_T w = w$ and $BSw = w$,

$$Aw = A(i_T w) = A(i_T BSw) = Sw,$$

showing that $S \subset A$.

Define $D_{\widetilde{T}} = D_A = i_T BV$ and $\widetilde{T} = A - 1$. Then $\widetilde{T}$ is self-adjoint, and for $w \in D_T$,

$$\widetilde{T}w = Aw - w = Sw - w = Tw,$$

showing that $T \subset \widetilde{T}$. For $x = i_T Bv$ we have obtained $\langle Ax, x\rangle \geq (Bv, Bv)_T$, and using this with $\|i_T\| \leq 1$ yields

$$\langle Ax, x\rangle \geq (Bv, Bv)_T = (i_T^{-1}x)_T^2 \geq |x|^2,$$

hence

$$\left\langle \widetilde{T}x, x \right\rangle = \langle Ax - x, x\rangle = \langle Ax, x\rangle - \langle x, x\rangle \geq 0,$$

showing that $\widetilde{T}$ is positive.

For $v \in D_T$ and $w \in V$,

$$\begin{aligned}\left\langle (1+T)v, (1+\widetilde{T})^{-1}w \right\rangle &= \left\langle Sv, A^{-1}w \right\rangle \\ &= \langle Sv, i_T Bw\rangle \\ &= \langle Av, i_T Bw\rangle \\ &= \left\langle v, A^{-1} i_T Bw \right\rangle \\ &= \langle v, w\rangle .\end{aligned}$$

We call the positive self-adjoint operator $\widetilde{T}, D_{\widetilde{T}}$ the **Friedrichs extension** of the positive densely defined operator T, D_T.[6]

Theorem 14 (Friedrichs extension theorem)**.** *If T, D_T is a positive densely defined operator then there is a positive self-adjoint extension $\widetilde{T}, D_{\widetilde{T}}$ of T that satisfies*

$$\left\langle (1+T)v, (1+\widetilde{T})^{-1}w \right\rangle = \langle v, w\rangle, \qquad v \in D_T, \quad w \in V.$$

[6] http://www.math.umn.edu/~garrett/m/v/friedrichs.pdf

Corollary 15. *If $i_T : V_T \to V$ is a compact operator, then $(1+\widetilde{T})^{-1} : V \to V$ is a compact operator.*

Proof. $\widetilde{T} = A - 1$ and $A = (i_T B)^{-1}$, so $(1+\widetilde{T})^{-1} = A^{-1} = i_T B$. Because $B : V \to V_T$ is continuous and $i_T : V_T \to V$ is compact, the composition $i_T B : V \to V$ is compact. □

4 The Laplacian on $L^2(\mathbb{T}^d)$

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and let m be the Haar probability measure on $\mathbb{T}^d$,

$$dm(x) = (2\pi)^{-d} dx.$$

Let $V = L^2(\mathbb{T}^d)$, let $D_T = C^\infty(\mathbb{T}^d)$, and let $T = -\Delta$. For $f, g \in D_T$,

$$\langle \Delta f, g \rangle = \int_{\mathbb{T}^d} \Delta f \cdot \overline{g} dm = -\int_{\mathbb{T}^d} \sum_{j=1}^d \partial_j f \cdot \overline{\partial_j g} dm = \langle f, \Delta g \rangle ,$$

showing that the densely defined operator T, D_T is symmetric, and

$$\langle Tf, f \rangle = -\langle \Delta f, f \rangle \geq 0,$$

showing that T is positive.

We follow the construction of the Friedrichs extension. For $f, g \in D_T$,

$$\begin{aligned}
(f,g)_T &= \langle f, g \rangle + \langle Tf, g \rangle \\
&= \langle f, g \rangle - \int_{\mathbb{T}^d} \Delta \cdot \overline{g} dm \\
&= \langle f, g \rangle + \int_{\mathbb{T}^d} \sum_{j=1}^d \partial_j f \cdot \overline{\partial_j g} dm \\
&= \langle f, g \rangle + \sum_{j=1}^d \langle \partial_j f, \partial_j g \rangle .
\end{aligned}$$

Then

$$(f)_T^2 = |f|^2 + \sum_{j=1}^d |\partial_j f|^2.$$

V_T is the completion of the inner product space $(D_T, (\cdot,\cdot)_T)$, and $i_T : V_T \to V$ is defined as follows: for $\phi \in V_T$ there is a $(\cdot)_T$ Cauchy sequence f_n in D_T that converges to ϕ in the norm $(\cdot)_T$. Then f_n is a $|\cdot|$ Cauchy sequence, and converges to $i_T\phi \in V$ in the norm $|\cdot|$.

By the Friedrichs extension theorem, there is a positive self-adjoint extension $\widetilde{T}, D_{\widetilde{T}}$ of T, D_T such that

$$\left\langle (1+T)f, (1+\widetilde{T})^{-1} g \right\rangle = \langle f, g \rangle , \qquad f \in C^\infty(\mathbb{T}^d), \quad g \in L^2(\mathbb{T}^d).$$