

# The Schwartz space and the Fourier transform

Jordan Bell

jordan.bell@gmail.com

Department of Mathematics, University of Toronto

August 12, 2015

## 1 Schwartz functions

Let  $\mathcal{S}(\mathbb{R}^n)$  be the collection of Schwartz functions  $\mathbb{R}^n \rightarrow \mathbb{C}$ . For  $p \geq 0$  and  $\phi \in \mathcal{S}$ , write

$$\|\phi\|_p^2 = \sum_{|\nu| \leq p} \int_{\mathbb{R}^n} (1 + |x|^2)^p |(D^\nu \phi)(x)|^2 dx.$$

With the metric

$$d(\phi, \psi) = \sum_{p \geq 0} 2^{-p} \frac{\|\phi - \psi\|_p}{1 + \|\phi - \psi\|_p},$$

$\mathcal{S}$  is a Fréchet space.

For a multi-index  $\alpha$  and for  $\phi \in \mathcal{S}$ ,  $x \mapsto x^\alpha \phi(x)$  belongs to  $\mathcal{S}$  and we define  $X^\alpha : \mathcal{S} \rightarrow \mathcal{S}$  by  $(X^\alpha \phi)(x) = x^\alpha \phi(x)$ .  $D^\alpha \phi \in \mathcal{S}$  and

$$\|D^\alpha \phi\|_p^2 = \sum_{|\nu| \leq p} \int_{\mathbb{R}^n} (1 + |x|^2)^p |(D^{\nu+\alpha} \phi)(x)|^2 dx \leq \|\phi\|_{p+|\alpha|}^2.$$

Because  $|\{\mu : |\mu| = k\}| = \binom{n+k-1}{k}$ ,<sup>1</sup>

$$|\{\mu : \mu \leq \nu\}| \leq |\{\mu : |\mu| \leq |\nu|\}| \leq \binom{n+|\nu|}{|\nu|}.$$

The product rule states

$$D^\nu (fg) = \sum_{\mu \leq \nu} \binom{\nu}{\mu} (D^\mu f)(D^{\nu-\mu} g),$$

---

<sup>1</sup>Arthur T. Benjamin and Jennifer J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, p. 71, Identity 143 and p. 74, Identity 149.

and with the Cauchy-Schwarz inequality we obtain for  $|\nu| \leq p$ ,

$$\begin{aligned} |D^\nu(X^\alpha \phi)|^2 &= \left| \sum_{\mu \leq \nu} \binom{\nu}{\mu} (D^\mu \phi)(D^{\nu-\mu} X^\alpha) \right|^2 \\ &\leq \binom{n+p}{p} \sum_{|\mu| \leq p} \binom{\nu}{\mu}^2 |D^\mu \phi|^2 |D^{\nu-\mu} X^\alpha|^2, \end{aligned}$$

and with this

$$\begin{aligned} \|X^\alpha \phi\|_p^2 &= \sum_{|\nu| \leq p} \int_{\mathbb{R}^n} (1+|x|^2)^p |(D^\nu(X^\alpha \phi))(x)|^2 dx \\ &\leq \sum_{|\nu| \leq p} \int_{\mathbb{R}^n} (1+|x|^2)^p \binom{n+p}{p} \sum_{|\mu| \leq p} \binom{\nu}{\mu}^2 |D^\mu \phi|^2 |D^{\nu-\mu} X^\alpha|^2 dx \\ &\leq C_p \|\phi\|_{p+|\alpha|}^2. \end{aligned}$$

For  $g, \phi \in \mathcal{S}$  we have  $g\phi \in \mathcal{S}$ , and using the product rule we get

$$\|g\phi\|_p^2 \leq C_{p,g} \|\phi\|_p^2.$$

Therefore,

$$\phi \mapsto D^\alpha \phi, \quad \phi \mapsto X^\alpha \phi, \quad \phi \mapsto g\phi$$

are continuous linear maps  $\mathcal{S} \rightarrow \mathcal{S}$ .

## 2 Tempered distributions

For  $u : \mathcal{S} \rightarrow \mathbb{C}$ , we write

$$\langle \phi, u \rangle = u(\phi).$$

$\mathcal{S}'$  denotes the dual space of  $\mathcal{S}$ , and the elements of  $\mathcal{S}'$  are called **tempered distributions**. We assign  $\mathcal{S}'$  the weak-\* topology, the coarsest topology on  $\mathcal{S}'$  such that for each  $\phi \in \mathcal{S}$  the map  $u \mapsto \langle \phi, u \rangle$  is continuous  $\mathcal{S}' \rightarrow \mathbb{C}$ .

For  $\psi \in \mathcal{S}$ , we define  $\Lambda_\psi : \mathcal{S} \rightarrow \mathbb{C}$  by

$$\langle \phi, \Lambda_\psi \rangle = \int_{\mathbb{R}^n} \phi(x) \psi(x) dx, \quad \phi \in \mathcal{S},$$

and by the Cauchy-Schwarz inequality,

$$|\langle \phi, \Lambda_\psi \rangle| \leq \left( \int_{\mathbb{R}^n} |\phi(x)|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^n} |\psi(x)|^2 dx \right)^{1/2} = \|\psi\|_0 \|\phi\|_0,$$

whence  $\Lambda_\psi \in \mathcal{S}'$ . It is apparent that  $\psi \mapsto \Lambda_\psi$  is linear. Suppose that  $\psi_i \rightarrow \psi$  in  $\mathcal{S}$ , and let  $\phi \in \mathcal{S}$ . Then

$$|\langle \phi, \Lambda_{\psi_i} \rangle - \langle \phi, \Lambda_\psi \rangle| = |\langle \phi, \Lambda_{\psi_i - \psi} \rangle| \leq \|\psi_i - \psi\|_0 \|\phi\|_0 \rightarrow 0,$$

which shows that  $\psi \mapsto \Lambda_\psi$  is continuous. If  $\Lambda_\psi = 0$ , then in particular  $\Lambda_\psi \bar{\psi} = 0$ , i.e.  $\int_{\mathbb{R}^n} |\psi(x)|^2 dx = 0$ , which implies that  $\psi(x) = 0$  for almost all  $x$  and because  $\psi$  is continuous,  $\psi = 0$ . Therefore,  $\psi \mapsto \Lambda_\psi$  is a continuous linear injection  $\mathcal{S} \rightarrow \mathcal{S}'$ . It can be proved that  $\Lambda(\mathcal{S})$  is dense in  $\mathcal{S}'$ .<sup>2</sup>

For a multi-index  $\alpha$  and  $u \in \mathcal{S}'$ , we define  $D^\alpha u : \mathcal{S} \rightarrow \mathbb{C}$  by

$$\langle \phi, D^\alpha u \rangle = (-1)^{|\alpha|} \langle D^\alpha \phi, u \rangle, \quad \phi \in \mathcal{S}.$$

For  $\phi_i \rightarrow \phi$  in  $\mathcal{S}$ , because  $D^\alpha : \mathcal{S} \rightarrow \mathcal{S}$  and  $u : \mathcal{S} \rightarrow \mathbb{C}$  are continuous,

$$\langle \phi_i, D^\alpha u \rangle = (-1)^{|\alpha|} \langle D^\alpha \phi_i, u \rangle \rightarrow (-1)^{|\alpha|} \langle D^\alpha \phi, u \rangle = \langle \phi, D^\alpha u \rangle,$$

and therefore  $D^\alpha u \in \mathcal{S}'$ .

We define  $X^\alpha u : \mathcal{S} \rightarrow \mathbb{C}$  by

$$\langle \phi, X^\alpha u \rangle = \langle X^\alpha \phi, u \rangle, \quad \phi \in \mathcal{S}.$$

For  $\phi_i \rightarrow \phi$  in  $\mathcal{S}$ ,

$$\langle \phi_i, X^\alpha u \rangle = \langle X^\alpha \phi_i, u \rangle \rightarrow \langle X^\alpha \phi, u \rangle = \langle \phi, X^\alpha u \rangle,$$

and therefore  $X^\alpha u \in \mathcal{S}'$ .

For  $g \in \mathcal{S}$ , we define  $gu : \mathcal{S} \rightarrow \mathbb{C}$  by

$$\langle \phi, gu \rangle = \langle g\phi, u \rangle, \quad \phi \in \mathcal{S}.$$

For  $\phi_i \rightarrow \phi$  in  $\mathcal{S}$ ,

$$\langle \phi_i, gu \rangle = \langle g\phi_i, u \rangle \rightarrow \langle g\phi, u \rangle = \langle \phi, gu \rangle,$$

and therefore  $gu \in \mathcal{S}'$ .

For  $\psi \in \mathcal{S}$ , integrating by parts yields

$$\begin{aligned} \langle \phi, D^\alpha \Lambda_\psi \rangle &= (-1)^{|\alpha|} \langle D^\alpha \phi, \Lambda_\psi \rangle \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} (D^\alpha \phi)(x) \psi(x) dx \\ &= \int_{\mathbb{R}^n} \phi(x) (D^\alpha \psi)(x) dx \\ &= \langle \phi, \Lambda_{D^\alpha \psi} \rangle, \end{aligned}$$

which implies that  $D^\alpha \Lambda_\psi = \Lambda_{D^\alpha \psi}$ .

$$\langle \phi, X^\alpha \Lambda_\psi \rangle = \langle X^\alpha \phi, \Lambda_\psi \rangle = \int_{\mathbb{R}^n} x^\alpha \phi(x) \psi(x) dx = \langle \phi, \Lambda_{X^\alpha \psi} \rangle,$$

which implies that  $X^\alpha \Lambda_\psi = \Lambda_{X^\alpha \psi}$ .

---

<sup>2</sup>Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics, volume I: Functional Analysis*, revised and enlarged edition, p. 144, Corollary 1 to Theorem V.14.

$$\langle \phi, g\Lambda_\psi \rangle = \langle g\phi, \Lambda_\psi \rangle = \int_{\mathbb{R}^n} g(x)\phi(x)\psi(x)dx = \langle \phi, \Lambda_{g\psi} \rangle,$$

which implies that  $g\Lambda_\psi = \Lambda_{g\psi}$ .

Because  $\phi \mapsto D^\alpha\phi$ ,  $\phi \mapsto X^\alpha\phi$ , and  $\phi \mapsto g\phi$  are continuous linear maps  $\mathcal{S} \rightarrow \mathcal{S}$  and because  $\Lambda : \mathcal{S} \rightarrow \mathcal{S}'$  is a continuous linear map with dense image, using the above it is proved that

$$u \mapsto D^\alpha u, \quad u \mapsto X^\alpha u, \quad u \mapsto gu$$

are continuous linear maps  $\mathcal{S}' \rightarrow \mathcal{S}'$ .<sup>3</sup>

### 3 The Fourier transform

For Borel measurable functions  $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ , for those  $x$  for which the integral exists we write

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} f(y)g(x-y)dy, \quad x \in \mathbb{R}^n,$$

and for those Borel measurable  $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$  for which the integral exists we write

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx.$$

For  $\xi \in \mathbb{R}^n$  we define

$$e_\xi(x) = e^{2\pi i \xi \cdot x}, \quad x \in \mathbb{R}^n,$$

and for  $\phi \in \mathcal{S}$  we calculate, integrating by parts,

$$(D^\alpha \phi) * e_\xi = (2\pi i \xi)^\alpha \phi * e_\xi.$$

We define  $\mathcal{F}\phi : \mathbb{R}^n \rightarrow \mathbb{C}$  by

$$(\mathcal{F}\phi)(\xi) = \langle \phi, e_\xi \rangle_{L^2} = \int_{\mathbb{R}^n} \phi(x)\overline{e_\xi(x)}dx = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} \phi(x)dx, \quad \xi \in \mathbb{R}^n,$$

which we can write as

$$(\phi * e_\xi)(0) = \int_{\mathbb{R}^n} \phi(y)e_\xi(-y)dy = \int_{\mathbb{R}^n} \phi(y)\overline{e_\xi(y)}dy = (\mathcal{F}\phi)(\xi).$$

By Fubini's theorem,

$$\begin{aligned} \mathcal{F}(\phi * \psi)(\xi) &= \int_{\mathbb{R}^n} \psi(y) \left( \int_{\mathbb{R}^n} \phi(x-y)\overline{e_\xi(x)}dx \right) dy \\ &= \int_{\mathbb{R}^n} \psi(y) \left( \int_{\mathbb{R}^n} \phi(x)\overline{e_\xi(x+y)}dx \right) dy, \end{aligned}$$

---

<sup>3</sup>Richard Melrose, *Introduction to Microlocal Analysis*, <http://math.mit.edu/~rbm/iml/Chapter1.pdf>, p. 17.

whence

$$\mathcal{F}(\phi * \psi) = (\mathcal{F}\phi)(\mathcal{F}\psi).$$

We calculate

$$\mathcal{F}(D^\alpha \phi)(\xi) = ((D^\alpha \phi) * e_\xi)(0) = ((2\pi i \xi)^\alpha \phi * e_\xi)(0) = (2\pi i \xi)^\alpha (\mathcal{F}\phi)(\xi),$$

whence

$$\mathcal{F}(D^\alpha \phi) = (2\pi i)^{|\alpha|} X^\alpha \mathcal{F}\phi.$$

It follows from the dominated convergence theorem

$$\begin{aligned} (D^\alpha \mathcal{F}\phi)(\xi) &= \int_{\mathbb{R}^n} (-2\pi i x)^\alpha e^{-2\pi i x \cdot \xi} \phi(x) dx \\ &= (-2\pi i)^{|\alpha|} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} x^\alpha \phi(x) dx \\ &= (-2\pi i)^{|\alpha|} \mathcal{F}(X^\alpha \phi)(\xi). \end{aligned}$$

Therefore

$$\mathcal{F}D^\alpha = (2\pi i)^{|\alpha|} X^\alpha \mathcal{F}, \quad D^\alpha \mathcal{F} = (-2\pi i)^{|\alpha|} \mathcal{F}X^\alpha. \quad (1)$$

Using the multinomial theorem,

$$\begin{aligned} (1 + |\xi|^2)^p |(D^\nu \mathcal{F}\phi)(\xi)|^2 &= \sum_{k=0}^p \binom{p}{k} |\xi|^{2k} |(D^\nu \mathcal{F}\phi)(\xi)|^2 \\ &= \sum_{k=0}^p \binom{p}{k} \sum_{|\alpha|=k} \binom{k}{\alpha} \xi^{2\alpha} |(D^\nu \mathcal{F}\phi)(\xi)|^2 \\ &= \sum_{k=0}^p \binom{p}{k} \sum_{|\alpha|=k} \binom{k}{\alpha} |(\xi^\alpha D^\nu \mathcal{F}\phi)(\xi)|^2. \end{aligned}$$

Applying (1),

$$|(\xi^\alpha D^\nu \mathcal{F}\phi)(\xi)| = (2\pi)^{|\nu|} (2\pi)^{-|\alpha|} |(\mathcal{F}D^\alpha X^\nu \phi)(\xi)|.$$

Then

$$\begin{aligned} \|\mathcal{F}\phi\|_p^2 &= \sum_{|\nu| \leq p} \int_{\mathbb{R}^n} (1 + |\xi|^2)^p |(D^\nu \mathcal{F}\phi)(\xi)|^2 d\xi \\ &= \sum_{|\nu| \leq p} \int_{\mathbb{R}^n} \sum_{k=0}^p \binom{p}{k} \sum_{|\alpha|=k} \binom{k}{\alpha} |(\xi^\alpha D^\nu \mathcal{F}\phi)(\xi)|^2 d\xi \\ &= \sum_{|\nu| \leq p} (2\pi)^{2|\nu|} \sum_{k=0}^p \binom{p}{k} (2\pi)^{-2k} \sum_{|\alpha|=k} \binom{k}{\alpha} \int_{\mathbb{R}^n} |(\mathcal{F}D^\alpha X^\nu \phi)(\xi)|^2 d\xi. \end{aligned}$$

Applying the Plancherel theorem, the product rule, and the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\int_{\mathbb{R}^n} |(\mathcal{F} D^\alpha X^\nu \phi)(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} |(D^\alpha X^\nu \phi)(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^n} \left| \sum_{\beta \leq \alpha} (D^\beta X^\nu)(D^{\alpha-\beta} \phi) \right|^2 d\xi \\
&\leq \int_{\mathbb{R}^n} \sum_{\beta \leq \alpha} |(D^\beta X^\nu)(\xi)|^2 \cdot \sum_{\beta \leq \alpha} |(D^{\alpha-\beta} \phi)(\xi)|^2.
\end{aligned}$$

This yields

$$\|\mathcal{F}\phi\|_p \leq C_p \|\phi\|_p,$$

whence  $\mathcal{F} : \mathcal{S} \rightarrow \mathcal{S}$  is continuous.

For  $p > n/2$ , using the Cauchy-Schwarz inequality and spherical coordinates<sup>4</sup> we calculate

$$\begin{aligned}
|(\mathcal{F}\phi)(\xi)| &\leq \int_{\mathbb{R}^n} (1+|x|^2)^{-p/2} (1+|x|^2)^{p/2} |\phi(x)| dx \\
&\leq \left( \int_{\mathbb{R}^n} (1+|x|^2)^{-p} dx \right)^{1/2} \left( \int_{\mathbb{R}^n} (1+|x|^2)^p |\phi(x)|^2 dx \right)^{1/2} \\
&= \left( \int_0^\infty \int_{S^{n-1}} (1+r^2)^{-p} d\sigma r^{n-1} dr \right)^{1/2} \left( \int_{\mathbb{R}^n} (1+|x|^2)^p |\phi(x)|^2 dx \right)^{1/2} \\
&= \left( \frac{\pi^{n/2} \Gamma(p - \frac{n}{2})}{\Gamma(p)} \right)^{1/2} \left( \int_{\mathbb{R}^n} (1+|x|^2)^p |\phi(x)|^2 dx \right)^{1/2} \\
&\leq \left( \frac{\pi^{n/2} \Gamma(p - \frac{n}{2})}{\Gamma(p)} \right)^{1/2} \|\phi\|_p.
\end{aligned}$$

---

<sup>4</sup><http://individual.utoronto.ca/jordanbell/notes/sphericalmeasure.pdf>