The Schrödinger kernel, spherical surface measure, Fourier restriction, and the Strichartz inequality

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1 The Schrödinger equation

For $\psi_0 \in \mathscr{S}(\mathbb{R}^d)$, suppose that $\psi : [0,\infty) \times \mathbb{R}^d \to \mathbb{C}$ satisfies

$$i\partial_t \psi + \Delta\psi = 0, \qquad \psi(0,x) = \psi_0(x), \qquad x \in \mathbb{R}^d. \tag{1}$$

We write

$$\widehat{\psi}(\xi) = \widehat{\psi}(0,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot x}\psi(x)dx.$$

Integrating by parts,

$$\widehat{\partial_j\psi}(\xi) = 2\pi i\xi_j \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot x}\psi(x)dx = 2\pi i\xi_j\widehat{\psi}(\xi)$$

and

$$\widehat{\partial_j^2\psi}(\xi) = -4\pi^2\xi_j^2 \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot x}\psi(x)dx = -4\pi^2\xi_j^2\widehat{\psi}(\xi)$$

and so

$$\widehat{\Delta\psi}(\xi) = -4\pi^2|\xi|^2\widehat{\psi}(\xi).$$

Then taking the Fourier transform of (1),

$$i\partial_t\widehat{\psi} - 4\pi^2|\xi|^2\widehat{\psi} = 0, \qquad \widehat{\psi}(0,\xi) = \widehat{\psi_0}(\xi), \qquad \xi \in \mathbb{R}^d.$$

For each $\xi \in \mathbb{R}^d$, the solution of the above initial value problem is

$$\widehat{\psi}(t,\xi) = e^{-4\pi^2 it|\xi|^2}\widehat{\psi_0}(\xi). \tag{2}$$

For $(t,\xi) \in [0,\infty) \times \mathbb{R}^d$, using the Fourier inversion theorem with (2),

$$\psi(t,x) = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x}\widehat{\psi}(t,\xi)d\xi = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x}e^{-4\pi^2 it|\xi|^2}\widehat{\psi_0}(\xi)d\xi.$$

Let

$$\mathcal{P} = \{(-2\pi|\xi|^2, \xi) : \xi \in \mathbb{R}^d\}.$$

For $F \in C_b(\mathbb{R}^{d+1})$, define

$$I(F) = \int_{\mathbb{R}^d} F(-2\pi|\xi|^2, \xi)\widehat{\psi_0}(\xi)d\xi,$$

which satisfies

$$|I(F)| \leqslant \|F\|_{L^\infty} \left\|\widehat{\psi_0}\right\|_{L^1}.$$

Therefore I is bounded linear functional on $C_0(\mathbb{R}^{d+1})$, so by the Riesz representation theorem there is a complex Borel measure μ on $\mathbb{R}^{d+1}$ such that

$$\int_{\mathbb{R}^d} F(-2\pi|\xi|^2, \xi)\widehat{\psi_0}(\xi)d\xi = \int_{\mathbb{R}^{d+1}} F d\mu$$

for all $F \in C_0(\mathbb{R}^{d+1})$. For $(t, x) \in [0, \infty) \times \mathbb{R}^d$, there is a sequence $F_n \in C_0(\mathbb{R}^{d+1})$ that tends to $F(\tau, \xi) = e^{2\pi i\xi\cdot x + 2\pi it\tau}$ in $C_b(\mathbb{R}^{d+1})$, and then

$$\begin{aligned}
\psi(t, x) &= \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x} e^{-4\pi^2 it|\xi|^2}\widehat{\psi_0}(\xi)d\xi \\
&= \int_{\mathbb{R}^d} F(-2\pi|\xi|^2, \xi)\widehat{\psi_0}(\xi)d\xi \\
&= \int_{\mathbb{R}^{d+1}} F d\mu \\
&= \int_{\mathbb{R}^{d+1}} e^{2\pi i\xi\cdot x + 2\pi it\tau} d\mu(\tau, \xi).
\end{aligned}$$

Theorem 1. *If ψ satisfies* (1) *then for $t \geqslant 0$ and $x \in \mathbb{R}^d$,*

$$\psi(t, x) = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x} e^{-4\pi^2 it|\xi|^2}\widehat{\psi_0}(\xi)d\xi = \int_{\mathbb{R}^{d+1}} e^{2\pi i\xi\cdot x + 2\pi it\tau} d\mu(\tau, \xi).$$

For an invertible real symmetric $d \times d$ matrix T, with index $\sigma = \nu_+ - \nu_-$, where ν_+ is the number of positive eigenvalues of T counted with multiplicity and likewise for ν_-, let

$$G_T(x) = e^{-\pi i\langle Tx, x\rangle}, \qquad x \in \mathbb{R}^d.$$

Because G_T is bounded and continuous, it is a tempered distribution. The Fourier transform of the tempered distribution is the following tempered distribution.[1]

Lemma 2.

$$\widehat{G_T} = e^{-\pi i\frac{\sigma}{4}}|\det T|^{-1/2} G_{-T^{-1}},$$

[1] Thomas H. Wolff, *Lectures on Harmonic Analysis*, p. 24, Proposition 4.2; Robert S. Strichartz, *A Guide to Distribution Theory and Fourier Transforms*, p. 50, Example 3.

so

$$G_T = e^{-\pi i \frac{\sigma}{4}} |\det T|^{-1/2} \widehat{G_{-T^{-1}}}$$

and

$$\widecheck{G_T} = e^{-\pi i \frac{\sigma}{4}} |\det T|^{-1/2} G_{-T^{-1}}.$$

For $T = 4\pi t I$ and $t > 0$, (2) reads

$$\widehat{\psi} = G_T \widehat{\psi_0}.$$

T is an invertible real symmetric $d \times d$ matrix, with $\sigma(T) = d$ and $\det T = (4\pi t)^d$, so

$$\widecheck{G_T} = e^{-\pi i \frac{d}{4}} (4\pi t)^{-d/2} G_{-T^{-1}},$$

and applying the inverse Fourier transform we obtain

$$\psi = \widecheck{G_T} * \psi_0 = e^{-\pi i \frac{d}{4}} (4\pi t)^{-d/2} G_{-T^{-1}} * \psi_0.$$

That is,

$$\begin{aligned}\psi(t,x) &= e^{-\pi i \frac{d}{4}} (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-\pi i \left\langle -T^{-1}(x-y), x-y \right\rangle} \psi_0(y) dy \\ &= e^{-\pi i \frac{d}{4}} (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{i \frac{|x-y|^2}{4t}} \psi_0(y) dy.\end{aligned}$$

Theorem 3. *If ψ satisfies* (1)*, then for $t > 0$ and $x \in \mathbb{R}^d$,*

$$\psi(t,x) = e^{-\pi i \frac{d}{4}} (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{i \frac{|x-y|^2}{4t}} \psi_0(y) dy.$$

Then

$$|\psi(t,x)| \leqslant (4\pi t)^{-d/2} \int_{\mathbb{R}^d} |\psi_0(y)| dy.$$

Corollary 4. *If ψ satisfies* (1)*, then for $t > 0$,*

$$\|\psi(t)\|_{L^\infty} \leqslant (4\pi t)^{-d/2} \|\psi_0\|_{L^1}.$$

For a Hilbert space H, let $\mathscr{U}(H)$ be the set of unitary operators $H \to H$. If A is a densely defined self-adjoint operator in H and $t \in \mathbb{R}$, we make sense of the expression e^{itA} using the spectral theorem, and $(e^{itA})_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group.[2] The infinitesimal generator of this one-parameter group is iA. Let $A = \Delta$ on $L^2(\mathbb{R}^d)$.[3] Then $(e^{it\Delta})' = i\Delta e^{it\Delta}$: for $\psi_0 \in \mathscr{D}(\Delta)$ and for $\psi(t,x) = (e^{it\Delta}\psi_0)(x)$,

$$(\partial_t \psi)(t,x) = (e^{it\Delta}\psi_0)'(x) = (i\Delta e^{it\Delta}\psi_0)(x) = (i\Delta\psi(t))(x) = i(\Delta\psi)(t,x).$$

Thus, for $\psi_0 \in \mathscr{S}(\mathbb{R}^d)$, the function $\psi(t) = e^{it\Delta}\psi_0$ satisfies the initial value problem (1).

[2] `http://individual.utoronto.ca/jordanbell/notes/trotter.pdf`, §6.

[3] cf. `http://individual.utoronto.ca/jordanbell/notes/laplaceoperator.pdf`

2 Spherical surface measure

Let σ be surface measure on the unit sphere S^{d-1} in $\mathbb{R}^d$.[4] It satisfies $\sigma(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$. Define $A : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with $A(x, y) = x + y$. For a Borel set E in $\mathbb{R}^d$, by Fubini's theorem,

$$\begin{aligned}(\sigma * \sigma)(E) &= A_*(\sigma \times \sigma)(E) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} 1_E(x + y) d(\sigma \times \sigma)(x, y) \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} 1_E(x + y) d\sigma(x) \right) d\sigma(y) \\ &= \int_{\mathbb{R}^d} \sigma(E - y) d\sigma(y).\end{aligned}$$

For $T \in SO(d)$, because σ is $SO(d)$-invariant,

$$T_*(\sigma * \sigma)(E) = \int_{\mathbb{R}^d} \sigma(T^{-1}(E) - y) d\sigma(y) = \int_{\mathbb{R}^d} \sigma(E - Ty) d\sigma(y) = \int_{\mathbb{R}^d} \sigma(E - y) d\sigma(y).$$

This shows that $\sigma * \sigma$ is $SO(d)$-invariant.

Assume that there is a function κ such that

$$d(\sigma * \sigma)(x) = \kappa(|x|) dx.$$

We then calculate

$$\begin{aligned}\int_0^\infty F(r^2)\kappa(r) r^{d-1} dr &= \frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} \int_0^\infty F(r^2)\kappa(r) r^{d-1} dr d\sigma \\ &= \frac{1}{\sigma(S^{d-1})} \int_{\mathbb{R}^d} F(|x|^2)\kappa(|x|) dx \\ &= \frac{1}{\sigma(S^{d-1})} \int_{\mathbb{R}^d} F(|x|^2) d(\sigma * \sigma)(x) \\ &= \frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} \int_{S^{d-1}} F(|x + y|^2) d\sigma(x) d\sigma(y) \\ &= \frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} \int_{S^{d-1}} F(|x + e_1|^2) d\sigma(x) d\sigma(y) \\ &= \frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} \int_{S^{d-1}} F(2 + 2\langle x, e_1 \rangle) d\sigma(x) d\sigma(y) \\ &= \int_{S^{d-1}} F(2 + 2x_1) d\sigma(x).\end{aligned}$$

[4] `http://individual.utoronto.ca/jordanbell/notes/harmonicpolynomials.pdf`, §2.

Now, using spherical coordinates on S^{d-1},[5]

$$\begin{aligned}&\int_{S^{d-1}} g(x)d\sigma(x)\\ =&\int_{\phi_1=0}^{\pi}\cdots\int_{\phi_{d-2}=0}^{\pi}\int_{\phi_{d-1}=0}^{2\pi} g(x(\phi))(\sin\phi_1)^{d-2}\cdots(\sin\phi_{d-3})^2(\sin\phi_{d-2})d\phi_{d-1}\cdots d\phi_1,\end{aligned}$$

where

$$\begin{aligned}x_1 &= \cos\phi_1\\ x_2 &= \sin\phi_1\cos\phi_2\\ &\cdots\\ x_{d-1} &= \sin\phi_1\sin\phi_2\cdots\sin\phi_{d-2}\cos\phi_{d-1}\\ x_d &= \sin\phi_1\sin\phi_2\cdots\sin\phi_{d-2}\sin\phi_{d-1}.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{S^{d-1}} F(2+2x_1)d\sigma(x)\\ =&\int_{\phi_1=0}^{\pi}\cdots\int_{\phi_{d-2}=0}^{\pi}\int_{\phi_{d-1}=0}^{2\pi} F(2+2\cos\phi_1)\\ &\cdot(\sin\phi_1)^{d-2}\cdots(\sin\phi_{d-3})^2(\sin\phi_{d-2})d\phi_{d-1}\cdots d\phi_1\\ =&c_d\int_{\phi_1=0}^{\pi} F(2+2\cos\phi_1)\cdot(\sin\phi_1)^{d-2}d\phi_1,\end{aligned}$$

where

$$\begin{aligned}c_d &= 2\pi\cdot\prod_{k=1}^{d-3}\int_{\phi=0}^{\pi}(\sin\phi)^k d\phi\\ &= 2\pi\cdot\prod_{k=1}^{d-3}\pi^{1/2}\frac{\Gamma\left(\frac{1}{2}+\frac{k}{2}\right)}{\Gamma\left(1+\frac{k}{2}\right)}\\ &= 2\pi\cdot\pi^{\frac{d-3}{2}}\cdot\frac{\Gamma(1)}{\Gamma\left(1+\frac{d-3}{2}\right)}\\ &= 2\pi^{\frac{d-1}{2}}\frac{1}{\Gamma\left(\frac{d-1}{2}\right)}.\end{aligned}$$

Now we have, doing the change of variable $t^2=2+2\cos\phi$, for which $(\sin\phi)^2=\frac{4t^2-t^4}{4}$,

$$\begin{aligned}\int_0^\infty F(r^2)\kappa(r)r^{d-1}dr &= 2\pi^{\frac{d-1}{2}}\frac{1}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^\pi F(2+2\cos\phi)\cdot(\sin\phi)^{d-2}d\phi\\ &= 2\pi^{\frac{d-1}{2}}\frac{1}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^2 F(t^2)\cdot t\left(\frac{4t^2-t^4}{4}\right)^{\frac{d-3}{2}}dt,\end{aligned}$$

[5] Loukas Grafakos, *Classical Fourier Analysis*, second ed., p. 441, Appendix D.1.

yielding the expression

$$\kappa(r)r^{d-1} = 2^{-d+4}\pi^{\frac{d-1}{2}}\frac{1}{\Gamma\left(\frac{d-1}{2}\right)}\chi_{[0,2]}(r)r(4r^2 - r^4)^{\frac{d-3}{2}},$$

that is,

$$\kappa(r) = 2^{-d+4}\pi^{\frac{d-1}{2}}\frac{1}{\Gamma\left(\frac{d-1}{2}\right)}\chi_{[0,2]}(r)r^{1-d+1+d-3}(4 - r^2)^{\frac{d-3}{2}}.$$

Therefore

$$\begin{aligned} d(\sigma * \sigma)(x) &= \kappa(|x|)dx \\ &= 2^{-d+4}\pi^{\frac{d-1}{2}}\frac{1}{\Gamma\left(\frac{d-1}{2}\right)}\chi_{[0,2]}(|x|)|x|^{-1}(4 - |x|^2)^{\frac{d-3}{2}}dx. \end{aligned}$$

Theorem 5. *$\sigma * \sigma$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$:*

$$d(\sigma * \sigma)(x) = 2^{-d+4}\pi^{\frac{d-1}{2}}\frac{1}{\Gamma\left(\frac{d-1}{2}\right)}\chi_{[0,2]}(|x|)|x|^{-1}(4 - |x|^2)^{\frac{d-3}{2}}dx.$$

We compute

$$\int_0^2 r^{-1}(4 - r^2)^{\frac{d-3}{2}} \cdot r^{d-1}dr = 2^{-3+d}\pi^{1/2}\frac{\Gamma\left(\frac{d}{2} - \frac{1}{2}\right)}{\Gamma(d/2)},$$

and thus using the above density of $\sigma * \sigma$ with respect to Lebesgue measure and polar coordinates,

$$\begin{aligned} &(\sigma * \sigma)(\mathbb{R}^d) \\ =&2^{-d+4}\pi^{\frac{d-1}{2}}\frac{1}{\Gamma\left(\frac{d-1}{2}\right)} \cdot \sigma(S^{d-1}) \cdot 2^{-3+d}\pi^{1/2}\frac{\Gamma\left(\frac{d}{2} - \frac{1}{2}\right)}{\Gamma(d/2)} \\ =&2\pi^{d/2}\sigma(S^{d-1})\frac{1}{\Gamma(d/2)} \\ =&\sigma(S^{d-1})^2. \end{aligned}$$

3 Radial functions

A function $f : \mathbb{R}^d \to \mathbb{C}$ is called **radial** if there is a function $f_0 : [0, \infty) \to \mathbb{C}$ such that $f(x) = f_0(|x|)$. The following is an expression for the Fourier transform of a radial function.[6]

[6] `http://individual.utoronto.ca/jordanbell/notes/sphericalmeasure.pdf`; Elias M. Stein and Guido Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, p. 155, Theorem 3.3.

Lemma 6. *For $f \in L^1$ and $\xi \in \mathbb{R}^d$,*

$$\widehat{f}(\xi) = 2\pi|\xi|^{-\frac{d}{2}+1} \int_0^\infty r^{d/2} J_{\frac{d}{2}-1}(2\pi r|\xi|) f_0(r) dr.$$

We prove conditions under which the Fourier transform of a radial function in L^p is continuous on $\mathbb{R}^d \backslash \{0\}$.[7]

Theorem 7. *If $f : \mathbb{R}^d \to \mathbb{C}$ is radial, with $f(x) = f_0(|x|)$, and $f \in L^p$ with $1 < p < \frac{2d}{d+1}$, then $\hat{f} \in C_0(\mathbb{R}^d \backslash \{0\})$ and*

$$\left\| \hat{f} \right\|_\infty \leqslant C_{d,p} \left\| f \right\|_{L^p}.$$

Proof. It is a fact that[8]

$$J_{\frac{d}{2}-1}(s) = (\pi s/2)^{-1/2} \cos\left(s - \frac{\pi d}{2} + \frac{\pi}{4}\right) + O(s^{-3/2}), \qquad s \to \infty,$$

and

$$J_{\frac{d}{2}-1}(s) \sim \frac{1}{\Gamma(d/2)} (s/2)^{\frac{d}{2}-1}, \qquad s \downarrow 0.$$

For $s > 0$ we get from the above asymptotic formulas

$$\frac{|J_{(d-2)/2}(s)|}{s^{(d-2)/2}} \leqslant C_d (1+s)^{-(d-1)/2}.$$

Using this, for $\xi \neq 0$ and $M > 0$,

$$\begin{aligned}
&\left| 2\pi|\xi|^{-\frac{d}{2}+1} \int_0^M r^{d/2} J_{\frac{d}{2}-1}(2\pi r|\xi|) f_0(r) dr \right| \\
=&2\pi \left| \int_0^M |\xi|^{-(d-2)/2} r^{-(d-2)/2} r^{(d-2)/2} r^{d/2} J_{(d-2)/2}(2\pi r|\xi|) f_0(r) dr \right| \\
\leqslant&2\pi \int_0^M |r\xi|^{-(d-2)/2} |J_{(d-2)/2}(2\pi r|\xi|)| |f_0(r)| r^{d-1} dr \\
\leqslant&2\pi \cdot C_d \cdot (2\pi)^{(d-2)/2} \cdot \int_0^M (1 + 2\pi r|\xi|)^{-(d-1)/2} |f_0(r)| r^{d-1} dr.
\end{aligned}$$

Using Hölder's inequality, for $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}
&\int_0^M (1 + 2\pi r|\xi|)^{-(d-1)/2} |f_0(r)| r^{(d-1)(1/p+1/q)} dr \\
\leqslant&\left(\int_0^M |f_0(r)|^p r^{d-1} dr \right)^{1/p} \left(\int_0^M (1 + 2\pi r|\xi|)^{-q(d-1)/2} r^{d-1} dr \right)^{1/q}.
\end{aligned}$$

[7] Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 157, Proposition 2.6.9.
[8] Elias M. Stein and Rami Shakarchi, *Complex Analysis*, p. 319, Appendix A.1.

If $-q(d-1)/2+d-1>-1$ then the second integral converges as $M\to\infty$, i.e. when $\frac{1}{q}>\frac{d-1}{2d}$, equivalently $\frac{1}{p}<\frac{d+1}{2d}$. For $\xi_k\to\xi\neq 0$, by the dominated convergence theorem,

$$\begin{aligned}&2\pi|\xi_k|^{-\frac{d}{2}+1}\int_0^\infty r^{d/2}J_{\frac{d}{2}-1}(2\pi r|\xi_k|)f_0(r)dr\\ \to&2\pi|\xi|^{-\frac{d}{2}+1}\int_0^\infty r^{d/2}J_{\frac{d}{2}-1}(2\pi r|\xi|)f_0(r)dr.\end{aligned}$$

Namely,

$$\begin{aligned}\widehat{f}(\xi)&=2\pi|\xi|^{-\frac{d}{2}+1}\int_0^\infty r^{d/2}J_{\frac{d}{2}-1}(2\pi r|\xi|)f_0(r)dr\\ &=2\pi\int_0^\infty(r|\xi|)^{-(d-2)/2}J_{(d-2)/2}(2\pi r|\xi|)f_0(r)r^{d-1}dr\end{aligned}$$

is continuous on $\mathbb{R}^d\backslash\{0\}$.

Let $\epsilon>0$. First,

$$\begin{aligned}&\left|2\pi\int_0^\epsilon(r|\xi|)^{-(d-2)/2}J_{(d-2)/2}(2\pi r|\xi|)f_0(r)r^{d-1}dr\right|\\ \leqslant&E_d\int_0^\epsilon|f_0(r)|r^{d-1}dr\\ =&E_d\int_0^\epsilon|f_0(r)|r^{(d-1)/p}r^{(d-1)/q}dr\\ \leqslant&E_d\left(\int_0^\epsilon|f_0(r)|^pr^{d-1}dr\right)^{1/p}\left(\int_0^\epsilon r^{d-1}dr\right)^{1/q}\\ =&E_d\frac{\epsilon^{d/q}}{d^{1/q}}\left(\int_0^\epsilon|f_0(r)|^pr^{d-1}dr\right)^{1/p}.\end{aligned}$$

Second,

$$\begin{aligned}&\left|2\pi\int_\epsilon^\infty(r|\xi|)^{-(d-2)/2}J_{(d-2)/2}(2\pi r|\xi|)f_0(r)r^{d-1}dr\right|\\ \leqslant&F_d\int_\epsilon^\infty(r|\xi|)^{-(d-2)/2}(2\pi r|\xi|)^{-1/2}|f_0(r)|r^{d-1}dr\\ =&F_d'|\xi|^{-(d-1)/2}\int_\epsilon^\infty r^{-(d-1)/2}|f_0(r)|r^{d-1}dr\\ =&F_d'|\xi|^{-(d-1)/2}\int_\epsilon^\infty r^{(d-1)(1/p+1/q)}r^{-(d-1)/2}|f_0(r)|dr\\ \leqslant&F_d'|\xi|^{-(d-1)/2}\left(\int_\epsilon^\infty|f_0(r)|^pr^{d-1}dr\right)^{1/p}\left(\int_\epsilon^\infty r^{d-1}r^{-q(d-1)/2}dr\right)^{1/q}\\ =&F_d'|\xi|^{-(d-1)/2}\left(\frac{\epsilon^{d-q(d-1)/2}}{q(d-1)/2-d}\right)^{1/q}\left(\int_\epsilon^\infty|f_0(r)|^pr^{d-1}dr\right)^{1/p}.\end{aligned}$$

□

4 The Tomas-Stein theorem

For $f \in L^p$, $p > 1$, the Fourier transform of f is a tempered distribution. The Hausdorff-Young inequality states that for $1 \leqslant p \leqslant 2$,

$$\left\| \widehat{f} \right\|_{L^q} \leqslant \|f\|_{L^p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. In fact, the Babenko-Beckner inequality[9] states that for $1 < p \leqslant 2$ and with $A_p = \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}}$,

$$\left\| \widehat{f} \right\|_{L^q} \leqslant A_p^d \|f\|_{L^p}.$$

If f and g are equal as elements of L^p, $1 \leqslant p \leqslant 2$, the Hausdorff-Young inequality tells us that their Fourier transforms are equal as elements of L^q, and so are equal almost everywhere with respect to Lebesgue measure. But σ is not absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$, so merely knowing that two functions are equal almost everywhere with respect to Lebesgue measure does not suffice for them to be equal almost everywhere with respect to σ. The **Tomas-Stein theorem** shows that this is the case when $1 \leqslant p \leqslant \frac{2d+2}{d+3}$.[10]

Theorem 8 (Tomas-Stein theorem). *For $1 \leqslant p \leqslant \frac{2d}{d+3}$, there is some $A_{d,p}$ such that for all $f \in \mathscr{S}(\mathbb{R}^d)$,*

$$\left(\int_{S^{d-1}} |\widehat{f}|^2 d\sigma \right)^{1/2} \leqslant A_{d,p} \|f\|_{L^p(\mathbb{R}^d)}.$$

We prove a subset of this theorem, for $1 \leqslant p < \frac{4d}{3d+1}$.[11]

Theorem 9 (Tomas theorem). *For $1 \leqslant p < \frac{4d}{3d+1}$ and for all $f \in \mathscr{S}(\mathbb{R}^d)$,*

$$\left(\int_{S^{d-1}} |\widehat{f}|^2 d\sigma \right)^{1/2} \leqslant \|\widehat{\sigma}\|_{L^{p/(2p-2)}}^{1/2} \cdot \|f\|_{L^p(\mathbb{R}^d)}.$$

Proof. The Fourier transform of σ is[12]

$$\widehat{\sigma}(\xi) = 2\pi|\xi|^{-\frac{d}{2}+1} J_{\frac{d}{2}-1}(2\pi|\xi|).$$

Thus,

$$\widehat{\sigma}(\xi) = O(|\xi|^{-(d-1)/2}), \qquad |\xi| \to \infty.$$

[9] D. J. H. Garling, *Inequalities: A Journey into Linear Analysis*, p. 231, Theorem 13.12.2.

[10] Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 160, Proposition 2.6.11; Camil Muscalu and Wilhelm Schlag, *Classical and Multilinear Harmonic Analysis*, volume I, p. 289, Lemma 11.2.

[11] Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 159, Proposition 2.6.10.

[12] `http://individual.utoronto.ca/jordanbell/notes/sphericalmeasure.pdf`

Because the Fourier transform of a complex Borel measure is continuous, using polar coordinates we check that $\widehat{\sigma} \in L^q(\mathbb{R}^d)$ if and only if $-q(d-1)/2+d-1 < -1$, i.e. $q > \frac{2d}{d-1}$; we shall write $\frac{1}{p} + \frac{1}{p'} = 1$.

Write

$$\widetilde{g}(x) = g(-x).$$

For a tempered distribution u and $\phi \in \mathscr{S}(\mathbb{R}^d)$, the convolution $u * \phi$ is defined by

$$(u * \phi)(x) = u(\tau_x \widetilde{\phi}), \qquad x \in \mathbb{R}^d,$$

and the Fourier transform of u is the tempered distribution defined by

$$\widehat{u}(\phi) = u(\widehat{\phi}).$$

It is a fact that if u is a tempered distribution then[13]

$$\widehat{\phi} * \widehat{u} = \widehat{\phi u}$$

and

$$(u * \phi) * \psi = u * (\phi * \psi).$$

Writing $Nf = \widetilde{f}$, for which we check $N(\widetilde{f} * f)(x) = (\widetilde{f} * f)(x)$, we get

$$\begin{aligned}
\int_{\mathbb{R}^d} |\widehat{f}|^2 d\sigma &= (\overline{f} * \widehat{\sigma})(f) \\
&= (\widetilde{f} * (f * \widehat{\sigma}))(0) \\
&= ((\widetilde{f} * f) * \widehat{\sigma})(0) \\
&= \widehat{\sigma}(N(\widetilde{f} * f)) \\
&= \widehat{\sigma}(\widetilde{f} * f) \\
&= \int_{\mathbb{R}^d} \widetilde{f} * f \cdot \widehat{\sigma} dx.
\end{aligned}$$

Therefore by Hölder's inequality with $\frac{1}{q} + \frac{1}{q'} = 1$ and then Young's inequality with $\frac{1}{r} + \frac{1}{r} = \frac{1}{q'} + 1$, we get

$$\int_{\mathbb{R}^d} |\widehat{f}|^2 d\sigma \leqslant \|\widehat{\sigma}\|_{L^q} \cdot \left\|\widetilde{f} * f\right\|_{L^{q'}} \leqslant \|\widehat{\sigma}\|_{L^q} \cdot \left\|\widetilde{f}\right\|_{L^r} \cdot \|f\|_{L^r} = \|\widehat{\sigma}\|_{L^q} \cdot \|f\|_{L^p}^2 .$$

For $r = p$ we have $\frac{2}{p} = 2 - \frac{1}{q}$, i.e. $q = \frac{p}{2p-2}$. Now from above, $\widehat{\mu} \in L^q$ if and only if $q > \frac{2d}{d-1}$, i.e. $\frac{1}{q} < \frac{d-1}{2d}$, which is equivalent to $\frac{2}{p} > \frac{3d+1}{2d}$, which is the case by hypothesis. □

[13] Walter Rudin, *Functional Analysis*, second ed., p. 195, Theorem 7.19.

5 The Hardy-Littlewood-Sobolev inequality

For $0 < \alpha < d$ and $f \in \mathscr{S}(\mathbb{R}^d)$, we define the **Riesz potential** I_α by

$$(I_\alpha f)(x) = \frac{1}{c_\alpha} \int_{\mathbb{R}^d} |x-y|^{-d+\alpha} f(y) dy,$$

where

$$c_\alpha = \pi^{d/2} 2^\alpha \frac{\Gamma(\alpha/2)}{\Gamma((d-\alpha)/2)}.$$

The Fourier transform of the tempered distribution $K_\alpha(x) = \frac{1}{c_\alpha}|x|^{-d+\alpha}$ is the tempered distribution[14]

$$\widehat{K}_\alpha(\xi) = |2\pi\xi|^{-\alpha}.$$

As $I_\alpha f = K_\alpha * f$, we have

$$\widehat{I_\alpha f}(\xi) = \widehat{K}_\alpha(\xi) \cdot \widehat{f}(\xi) = |2\pi\xi|^{-\alpha} \cdot \widehat{f}(\xi).$$

If $0 < \alpha, \beta, \alpha + \beta < d$ then from the above formula we get

$$I_\alpha I_\beta = I_{\alpha+\beta}.$$

Using $\Delta(I_\alpha f) = \Delta(K_\alpha * f) = K_\alpha * (\Delta f)$,[15] we calculate for $2 < \alpha < d$, [16]

$$\begin{aligned}
\widehat{\Delta(I_\alpha f)}(\xi) &= \widehat{I_\alpha(\Delta f)}(\xi) \\
&= |2\pi\xi|^{-\alpha} \cdot \widehat{\Delta f}(\xi) \\
&= |2\pi\xi|^{-\alpha} \cdot (-4\pi^2)|\xi|^2 \cdot \widehat{f}(\xi) \\
&= -|2\pi\xi|^{-\alpha+2} \cdot \widehat{f}(\xi) \\
&= -\widehat{I_{\alpha-2} f}(\xi).
\end{aligned}$$

This implies that $\Delta I_\alpha f = -I_{\alpha-2} f$.

Theorem 10. *For $0 < \alpha < d$ and for $f \in \mathscr{S}(\mathbb{R}^d)$,*

$$\widehat{I_\alpha f}(\xi) = |2\pi\xi|^{-\alpha} \cdot \widehat{f}(\xi).$$

I_α satisfies, for $0 < \alpha, \beta, \alpha + \beta < d$,

$$I_\alpha I_\beta = I_{\alpha+\beta}$$

and for $2 < \alpha < d$,

$$\Delta I_\alpha f = -I_{\alpha-2} f.$$

[14] Loukas Grafakos, *Classical Fourier Analysis*, second ed., p. 128, Theorem 2.4.6.

[15] Walter Rudin, *Functional Analysis*, second ed., p. 195, Theorem 7.19.

[16] cf. `http://individual.utoronto.ca/jordanbell/notes/dirac.pdf`

The following is the **Hardy-Littlewood-Sobolev inequality**.[17]

Theorem 11 (Hardy-Littlewood-Sobolev inequality)**.** *Suppose that* $0 < \alpha < d$ *and* $1 \leqslant p < q < \infty$ *with* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$*. There is* $C(d, \alpha, p) < \infty$ *such that*

$$\|I_\alpha f\|_{L^q} \leqslant C(d, \alpha, p) \|f\|_{L^p}$$

when $p > 1$*, and*

$$m(x \in \mathbb{R}^d : |I_\alpha f(x)| > \lambda)^{1/q} \leqslant C(d, \alpha, p) \frac{\|f\|_{L^1}}{\lambda}, \qquad \lambda > 0,$$

when $p = 1$*.*

6 The Strichartz inequality

We now prove a case of the **Strichartz inequality**, when $p = q$.[18]

Theorem 12 (Strichartz inequality)**.** *Suppose that* $\{U(t)\}_{t \in \mathbb{R}}$ *is a family of linear operators on* $L^1(\mathbb{R}^d) + L^2(\mathbb{R}^d)$ *and that there are* $C, \gamma > 0$ *such that*

$$\|U(t)U(s)^*\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leqslant C$$

and

$$\|U(t)U(s)^*\|_{L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)} \leqslant C|t - s|^{-\sigma}.$$

When

$$\frac{2}{p} + \frac{2\sigma}{q} = \sigma, \qquad 2 \leqslant p \leqslant \infty, \qquad 1 \leqslant q \leqslant \infty, \qquad (p, q) \neq (2, \infty),$$

it holds that

$$\left(\int_{\mathbb{R}} \|U(t)f\|_{L^q(\mathbb{R}^d)}^p \, dt \right)^{1/p} \leqslant C \|f\|_{L^2(\mathbb{R}^d)}.$$

Proof. For $p = q$, we wish to prove that

$$\|(U(t)f)(t, x)\|_{L^p(\mathbb{R} \times \mathbb{R}^d)} \leqslant C \|f\|_{L^2(\mathbb{R}^d)} \tag{3}$$

with $p = \frac{2(1+\sigma)}{\sigma}$.

Let $s, t \in \mathbb{R}$ and define $T = U(t)U(s)^*$. For $p_0 = 2$, $q_0 = 2$, $p_1 = 1$, $q_1 = \infty$, for $0 < r < 1$ let

$$\frac{1}{p_r} = \frac{1 - r}{p_0} + \frac{r}{p_1}, \qquad \frac{1}{q_r} = \frac{1 - r}{q_0} + \frac{r}{q_1},$$

[17] Loukas Grafakos, *Modern Fourier Analysis*, second ed., p. 3, Theorem 6.1.3; Elias M. Stein, *Singular Integrals and Differentiability Properties of Functions*, p. 119, Theorem 1; Elliott H. Lieb and Michael Loss, *Analysis*, second ed., p. 106, Theorem 4.3.

[18] Maciej Zworski, *Semiclassical Analysis*, p. 236, Theorem 10.7.

that is, $p_r = \frac{2}{1+r}$ and $q_r = \frac{2}{1-r}$. To have $q_r = p$ is equivalent to

$$r = 1 - \frac{2}{p} = \frac{1}{1+\sigma}.$$

Because $\|T\|_{L^{p_0}\to L^{q_0}} \leqslant C$ and $\|T\|_{L^{p_1}\to L^{q_1}} \leqslant C|t-s|^{-\sigma}$, by the Riesz-Thorin interpolation theorem,[19]

$$\|T\|_{L^{p_r}\to L^{q_r}} \leqslant C^{1-r}C^r|t-s|^{-\sigma r} = C|t-s|^{-\sigma\left(1-\frac{2}{p}\right)}.$$

For functions $F(t,x)$ and $G(t,x)$, using Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$,

$$\begin{aligned} |\langle U(t)^*G(t,\cdot), U(s)^*F(s,\cdot)\rangle| &\leqslant \|G(t,\cdot)\|_{L^{p'}} \cdot \|U(t)U(s)^*F(s,\cdot)\|_{L^p} \\ &\leqslant \|G(t,\cdot)\|_{L^{p'}} \cdot C|t-s|^{-\sigma\left(1-2+\frac{2}{p'}\right)} \cdot \|F(s,\cdot)\|_{L^{p_r}} \\ &= \|F(s,\cdot)\|_{L^{p'}} \cdot \|G(t,\cdot)\|_{L^{p'}} \cdot C|t-s|^{-\sigma\left(-1+\frac{2}{p'}\right)}. \end{aligned}$$

For $-1+\alpha = -\sigma\left(-1+\frac{2}{p'}\right)$, applying Hölder's inequality,

$$\begin{aligned} &\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \langle U(t)^*G(t,\cdot), U(s)^*F(s,\cdot)\rangle\, dtds\right| \\ \leqslant & C\int_{\mathbb{R}}\int_{\mathbb{R}} |t-s|^{-\sigma\left(-1+\frac{2}{p'}\right)} \cdot \|F(s,\cdot)\|_{L^{p'}} \cdot \|G(t,\cdot)\|_{L^{p'}}\, dtds \\ =& C\cdot c_\alpha \int_{\mathbb{R}} \left(\frac{1}{c_\alpha}\int_{\mathbb{R}} |t-s|^{-1+\alpha}\, \|G(t,\cdot)\|_{L^{p'}}\, dt\right) \|F(s,\cdot)\|_{L^{p'}}\, dt \\ =& C\cdot c_\alpha \int_{\mathbb{R}} I_\alpha(\|G(t,\cdot)\|_{L^{p'}})(s)\, \|F(s,\cdot)\|_{L^{p'}}\, ds \\ \leqslant & C\cdot c_\alpha \left(\int_{\mathbb{R}} I_\alpha(\|G(t,\cdot)\|_{L^{p'}})(s)^p ds\right)^{1/p} \left(\int_{\mathbb{R}} \|F(s,\cdot)\|_{L^{p'}}^{p'}\, ds\right)^{1/p'}. \end{aligned}$$

By the Hardy-Littlewood-Sobolev inequality, when $0<\alpha<1$ and $\frac{1}{p} = \frac{1}{\beta} - \alpha$, for which $\beta = p'$,

$$\left(\int_{\mathbb{R}} I_\alpha(\|G(t,\cdot)\|_{L^{p'}})(s)^p ds\right)^{1/p} \leqslant C(\alpha,p)\left(\int_{\mathbb{R}} \|G(s,\cdot)\|_{L^{p'}}^{p'}\, ds\right)^{1/p'}.$$

Therefore we have

$$\begin{aligned} &\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \langle U(t)^*G(t,\cdot), U(s)^*F(s,\cdot)\rangle\, dtds\right| \\ \leqslant & C\cdot c_\alpha \|F\|_{L^{p'}(\mathbb{R}\times\mathbb{R}^d)} \cdot \|G\|_{L^{p'}(\mathbb{R}\times\mathbb{R}^d)}. \end{aligned}$$

□

[19] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, second ed., p. 200, Theorem 6.27.