# Unordered sums in Hilbert spaces

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## 1 Preliminaries

Let $\mathbb{N}$ be the set of positive integers. We say that a set is countable if it is bijective with a subset of $\mathbb{N}$; thus a finite set is countable. In this note I do not presume unless I say so that any set is countable or that any topological space is separable. A neighborhood of a point in a topological space is a set that contains an open set that contains the point; one reason why it can be handy to speak about neighborhoods of a point rather than just open sets that contain the point is that the set of all neighborhoods of a point is a filter, whereas it is unlikely that the set of all open sets that contain a point is a filter.

## 2 Unordered sums in normed spaces

A *partially ordered set* is a set $J$ and a binary relation $\leq$ on $J$ that is reflexive ($\alpha \leq \alpha$), antisymmetric (if both $\alpha \leq \beta$ and $\beta \leq \alpha$ then $\alpha = \beta$), and transitive (if both $\alpha \leq \beta$ and $\beta \leq \gamma$ then $\alpha \leq \gamma$).[1] A *directed set* is a partially ordered set $(J, \leq)$ such that if $\alpha, \beta \in J$ then there is some $\gamma \in J$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. If $X$ is a topological space, a *net* in $X$ is a function from some directed set to $X$. If $z : J \to X$ is a net in $X$ and $N$ is a subset of $X$, we say that $z$ is *eventually in* $N$ if there is some $\alpha \in J$ such that $\alpha \leq \beta$ implies $z(\beta) \in N$. We say that the net $z$ *converges* to $x \in X$ if for every neighborhood of $x$ the net is eventually in that neighborhood. The importance of the notion of a net is that if $X$ and $Y$ are topological spaces and $f$ is a function $X \to Y$ then $f$ is continuous if and only if for every $x \in X$ and for every net $z : J \to X$ that converges to $x$, the net $f \circ z : J \to Y$ converges to $f(x)$.[2]

Let $X$ be a normed space, let $I$ be a set, and let $\mathscr{F}$ be the set of all finite subsets of $I$. $\mathscr{F}$ is a directed set ordered by set inclusion. Define $S : \mathscr{F} \to X$ by

$$S(F) = \sum_{i \in F} f(i) \in X, \qquad F \in \mathscr{F}.$$

[1] Paul R. Halmos, *Naive Set Theory*, §14.
[2] James R. Munkres, *Topology*, second ed., p. 188.

$S$ is a net in $X$, and if the net $S$ converges to $x \in X$, we say that the sum $\sum_{i \in I} f(i)$ *converges to* $x$, and write $\sum_{i \in I} f(i) = x$.

**Theorem 1.** *If $X$ is a normed space, $f : I \to X$ is a function, $x \in X$, and $I_0$ is a subset of $I$ such that if $i \in I \setminus I_0$ then $f(i) = 0$, then $\sum_{i \in I} f(i)$ converges to $x$ if and only if $\sum_{i \in I_0} f(i)$ converges to $x$.*

*Proof.* Let $\mathscr{F}$ be the set of all finite subsets of $I$, let $\mathscr{F}_0$ be the set of all finite subsets of $I_0$, define $S : \mathscr{F} \to X$ by $S(F) = \sum_{i \in F} f(i)$, and let $S_0$ be the restriction of $S$ to $\mathscr{F}_0$. Suppose that $\sum_{i \in I} f(i)$ converges to $x$, and let $\epsilon > 0$. There is some $F_\epsilon \in \mathscr{F}$ such that if $F_\epsilon \subseteq F \in \mathscr{F}$ then $\|S(F) - x\| < \epsilon$. Let $G_\epsilon = F_\epsilon \cap I_0$. If $G_\epsilon \subseteq G \in \mathscr{F}_0$, then

$$S_0(G) - x = \sum_{i \in G} f(i) - x = \sum_{i \in F} f(i) - x = S(F) - x,$$

giving $\|S_0(G) - x\| = \|S(F) - x\|$. Hence $G_\epsilon \subseteq G \in \mathscr{F}_0$ implies that $\|S_0(G) - x\| < \epsilon$, showing that the net $S_0$ converges to $x$, i.e. that $\sum_{i \in I_0} f(i)$ converges to $x$.

Suppose that $\sum_{i \in I_0} f(i)$ converges to $x$, and let $\epsilon > 0$. There is some $G_\epsilon \in \mathscr{F}_0$ such that if $G_\epsilon \subseteq G \in \mathscr{F}_0$ then $\|S_0(G) - x\| < \epsilon$. If $G_\epsilon \subseteq F \in \mathscr{F}$, then, with $G = F \cap I_0$,

$$S(F) - x = \sum_{i \in F} f(i) - x = \sum_{i \in G} f(i) - x = S_0(G) - x,$$

so $G_\epsilon \subseteq F \in \mathscr{F}$ implies that $\|S(F) - x\| < \epsilon$. This shows that $S$ converges to $x$, that is, that $\sum_{i \in I} f(i)$ converges to $x$. □

**Theorem 2.** *If $X$ is a normed space, $f : I \to X$ is a function, and $\sum_{i \in I} f(i)$ converges, then $\{i \in I : f(i) \neq 0\}$ is countable.*

*Proof.* Suppose that $\sum_{i \in I} f(i)$ converges to $x$, let $\mathscr{F}$ be the set of all finite subsets of $I$, and let $S(F) = \sum_{i \in I} f(i)$, $F \in \mathscr{F}$. For each $n \in \mathbb{N}$, let $F_n \in \mathscr{F}$ be such that if $F_n \subseteq F \in \mathscr{F}$ then

$$\|S(F) - x\| < \frac{1}{n}.$$

If $G \in \mathscr{F}$ and $G \cap F_n = \emptyset$, then

$$\|S(G)\| = \|S(G \cup F_n) - S(F_n)\| \leq \|S(G \cup F_n) - x\| + \|S(F_n) - x\| < \frac{2}{n}.$$

Let $J = \bigcup_{n \in \mathbb{N}} F_n$. If $i \in I \setminus J$, then for each $n \in \mathbb{N}$, we have $\{i\} \cap F_n = \emptyset$, whence $\|S(\{i\})\| < \frac{2}{n}$. That is, if $i \in I \setminus J$ then for each $n \in \mathbb{N}$ we have $\|f(i)\| < \frac{2}{n}$, which implies that if $i \in I \setminus J$ then $f(i) = 0$. Therefore $\{i \in I : f(i) \neq 0\} \subseteq J$, and as $J$ is countable, the set $\{i \in I : f(i) \neq 0\}$ is countable. □

However, we already have a notion of infinite sums: a series is the limit of a sequence of partial sums.

**Theorem 3.** *If $X$ is a normed space, $x_n \in X$, and $\sum_{n\in\mathbb{N}} x_n$ converges to $x$, then $\sum_{n=1}^{N} x_n \to x$ as $N \to \infty$.*

*Proof.* Let $\epsilon > 0$, let $\mathscr{F}$ be the set of all finite subsets of $\mathbb{N}$, and let $S : \mathscr{F} \to X$ be $S(F) = \sum_{n\in F} x_n$. The net $S$ converges to $x$, so there is some $F_\epsilon \in \mathscr{F}$ such that if $F_\epsilon \subseteq F$ then $\|S(F) - x\| < \epsilon$. Let $N_\epsilon = \max F_\epsilon$. If $N \geq N_\epsilon$, then for $F = \{1, \ldots, N\}$ we have $F_\epsilon \subseteq F$ and so

$$\left\| \sum_{n=1}^{N} x_n - x \right\| = \|S(F) - x\| < \epsilon,$$

showing that $\sum_{n=1}^{N} x_n \to x$ as $N \to \infty$. □

When we talk about the sum $\sum_{i\in I} f(i)$, the set of all finite subsets of $I$ is ordered by set inclusion, but we don't care about any ordering of the set $I$ itself. If the sum $\sum_{n\in\mathbb{N}} x_n$ converges then for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$, $\sum_{n=1}^{\infty} x_{\sigma(n)} = \sum_{n\in\mathbb{N}} x_n$. If $x_n$ is a sequence in a normed space and for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges, we say that the sequence $x_n$ is *unconditionally summable*. If an unordered sum converges, then it is unconditionally summable, and if a countable unordered sum is unconditionally summable the unordered sum converges.

**Theorem 4.** *If $X$ is a Banach space, $x_n \in X$, and $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $\sum_{n\in\mathbb{N}} x_n$ converges.*

*Proof.* For each $k \in \mathbb{N}$ there is some $K(k)$ such that

$$\sum_{n=K(k)+1}^{\infty} \|x_n\| < \frac{1}{k};$$

suppose that if $j < k$ then $K(j) < K(k)$. Define

$$v_k = \sum_{n=1}^{K(k)} x_n.$$

For $\epsilon > 0$, let $N > \frac{1}{\epsilon}$. If $k > j \geq N$, then

$$\|v_k - v_j\| = \left\| \sum_{n=1}^{K(k)} x_n - \sum_{n=1}^{K(j)} x_n \right\| = \left\| \sum_{n=K(j)+1}^{K(k)} x_n \right\| \leq \sum_{n=K(j)+1}^{K(k)} \|x_n\| \leq \sum_{n=K(j)+1}^{\infty} \|x_n\|,$$

hence if $k > j \geq N$, then $\|v_k - v_j\| < \frac{1}{j} \leq \frac{1}{N}$. This shows that $v_k$ is a Cauchy sequence, and hence $v_k$ converges to some $x \in X$.

Let $\mathscr{F}$ be the set of all finite subsets of $\mathbb{N}$ and define $S : \mathscr{F} \to X$ by $S(F) = \sum_{n\in F} x_n$. Let $\epsilon > 0$, and as $v_k \to x$ there is some $N_1$ such that if

$k \geq N_1$ then $\|v_k - x\| < \epsilon$. Let $N_2 > \frac{1}{\epsilon}$, put $N = \max\{N_1, N_2\}$, and put $F_\epsilon = \{1, \ldots, K(N)\}$. If $F_\epsilon \subseteq F \in \mathscr{F}$, then

$$\begin{aligned}
\|S(F) - x\| &= \left\| \sum_{n \in F} x_n - x \right\| \\
&\leq \left\| \sum_{n \in F} x_n - \sum_{n \in F_\epsilon} x_n \right\| + \left\| \sum_{n \in F_\epsilon} x_n - x \right\| \\
&= \left\| \sum_{n \in F \setminus F_\epsilon} x_n \right\| + \|v_N - x\| \\
&< \sum_{n \in F \setminus F_\epsilon} \|x_n\| + \epsilon \\
&\leq \sum_{n = K(N)+1}^{\infty} \|x_n\| + \epsilon \\
&< \frac{1}{N} + \epsilon \\
&< 2\epsilon.
\end{aligned}$$

Therefore the net $S$ converges to $x$, i.e. $\sum_{n \in \mathbb{N}} x_n$ converges to $x$. □

The following theorem shows us in particular that the converse of Theorem 3 is false. One direction of the following theorem is Theorem 4 with $X = \mathbb{C}$. The other direction follows from the Riemann rearrangement theorem.[3]

**Theorem 5.** *If $\alpha_n \in \mathbb{C}$, then $\sum_{n \in \mathbb{N}} \alpha_n$ converges if and only if $\sum_{n=1}^{\infty} |\alpha_n| < \infty$.*

Let $X$ be a normed space and $z : J \to X$ a net. We say that $z$ is *Cauchy* if for every $\epsilon > 0$ there is some $\alpha \in J$ such that $\alpha \leq \beta$ and $\alpha \leq \gamma$ together imply that $\|z(\beta) - z(\gamma)\| < \epsilon$.[4]

**Theorem 6.** *If $X$ is a Banach space and $z : J \to X$ is a Cauchy net, then there is some $x \in X$ such that $z$ converges to $x$.*

*Proof.* Let $\alpha_1 \in J$ such that if $\alpha_1 \leq \alpha$ then $\|z(\alpha) - z(\alpha_1)\| < 1$, and for $n > 1$ let $\alpha_n \in J$ be such that if $\alpha_n \leq \alpha$ then $\|z(\alpha) - z(\alpha_n)\| < \frac{1}{n}$ and such that $\alpha_{n-1} \leq \alpha_n$. Define $x_n = z(\alpha_n)$. For $\epsilon > 0$, let $N > \frac{1}{\epsilon}$. If $n \geq m \geq N$, then, as $\alpha_n \geq \alpha_m$,

$$\|x_n - x_m\| = \|z(\alpha_n) - z(\alpha_m)\| < \frac{1}{m} \leq \frac{1}{N},$$

showing that $x_n$ is a Cauchy sequence in $X$. Hence there is some $x \in X$ such that $x_n \to x$.

[3] Walter Rudin, *Principles of Mathematical Analysis*, third ed., p. 76, Theorem 3.54.

[4] Ronald G. Douglas, *Banach Algebra Techniques in Operator Theory*, second ed., p. 3, Proposition 1.7.

Let $\epsilon > 0$, let $N_1 > \frac{1}{\epsilon}$, let $N_2$ be such that if $n \geq N_2$ then $\|x_{N_2} - x\| < \epsilon$, and set $N = \max\{N_1, N_2\}$. If $\alpha_N \leq \alpha$, then, by construction of the sequence $\alpha_n$,

$$\begin{aligned} \|z(\alpha) - x\| &\leq \|z(\alpha) - z(\alpha_N)\| + \|z(\alpha_N) - x\| \\ &= \|z(\alpha) - z(\alpha_N)\| + \|x_N - x\| \\ &< \frac{1}{N} + \epsilon \\ &< 2\epsilon, \end{aligned}$$

showing that the net $z$ converges to $x$. □

**Theorem 7.** *If $H$ is an infinite dimensional Hilbert space and $\{e_n : n \in \mathbb{N}\}$ is an orthonormal set in $H$, then $\sum_{n\in\mathbb{N}} \frac{1}{n} e_n$ converges.*

*Proof.* Let $\mathscr{F}$ be the set of finite subsets of $\mathbb{N}$ and let $S(F) = \sum_{n\in F} \frac{1}{n} e_n$, $F \in \mathscr{F}$. Define $v_N = \sum_{n=1}^{N} \frac{1}{n} e_n$. If $N_1 > N_2 \geq N$, then, as $e_n$ are orthonormal,

$$\|v_{N_1} - v_{N_2}\|^2 = \left\| \sum_{n=N_2+1}^{N_1} \frac{1}{n} e_n \right\|^2 = \sum_{n=N_2+1}^{N_1} \frac{1}{n^2} < \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \sum_{n=N}^{\infty} \frac{1}{n(n+1)} = \frac{1}{N},$$

so $v_N$ is a Cauchy sequence in $H$ and hence converges to some $h \in H$. For $\epsilon > 0$, let $N_1 > \frac{1}{\epsilon}$, let $\|v_{N_2} - h\|^2 < \epsilon$, put $N = \max\{N_1, N_2\}$, and put $F_\epsilon = \{1, \ldots, N\}$. If $F_\epsilon \subseteq F \in \mathscr{F}$, then, using that $e_n$ are orthonormal and $0 \leq (a-b)^2 = a^2 - 2ab + b^2$,

$$\begin{aligned} \|S(F) - h\|^2 &\leq (\|S(F) - S(F_\epsilon)\| + \|S(F_\epsilon) - h\|)^2 \\ &\leq 2\|S(F) - S(F_\epsilon)\|^2 + 2\|S(F_\epsilon) - h\|^2 \\ &= 2\left\| \sum_{n\in F\setminus F_\epsilon} \frac{1}{n} e_n \right\|^2 + 2\|v_N - h\|^2 \\ &= 2\sum_{n\in F\setminus F_\epsilon} \frac{1}{n^2} + 2\|v_N - h\|^2 \\ &< 4\epsilon. \end{aligned}$$

This shows that the net $S$ converges to $h$, that is, that $\sum_{n\in\mathbb{N}} \frac{1}{n} e_n$ converges to $h$. □

We have proved that if $H$ is an infinite dimensional Hilbert space and $\{e_n : n \in \mathbb{N}\}$ is an orthonormal set in $H$, then $\sum_{n\in\mathbb{N}} \frac{1}{n} e_n$ converges, although $\sum_{n=1}^{\infty} \left\| \frac{1}{n} e_n \right\| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. This shows that the converse of Theorem 4 is false. In fact, the Dvoretsky-Rogers theorem states that if $X$ is an infinite dimensional Banach space then there is some countable subset $\{x_n : n \in \mathbb{N}\}$ of $X$ such that $\sum_{n\in\mathbb{N}} x_n$ converges but $\sum_{n\in\mathbb{N}} \|x_n\| = \infty$.[5]

[5] Joseph Diestel, *Sequences and Series in Banach Spaces*, p. 59, chapter VI.

## 3 Orthogonal projections

If $S_i, i \in I$, are subsets of a Hilbert space $H$, we define $\bigvee_{i\in I} S_i$ to be the closure of the span of $\bigcup_{i\in I} S_i$. If $i \neq j$ implies that $S_i \perp S_j$, we say that the sets $S_i$ are *mutually orthogonal*. To say that $\{e_i : i \in I\}$ is an orthonormal basis for $H$ is to say that $\{e_i : i \in I\}$ is an orthonormal set and that $H = \bigvee_{i\in I}\{e_i\}$.

If $M_n, n \in \mathbb{N}$, are mutually orthogonal closed subspaces of $M$, we denote

$$\bigoplus_{n\in\mathbb{N}} M_n = \bigvee_{n\in\mathbb{N}} M_n,$$

which we call an *orthogonal direct sum*.

If $H$ is a Hilbert space and $M$ is a closed subspace of $H$, then for every $h \in H$ there is a unique $v_h \in M$ such that

$$\|h - v_h\| = \inf_{v\in M} \|h - v\|,$$

and $h - v_h \in M^\perp$.[6] This gives

$$H = M \oplus M^\perp.$$

The *orthogonal projection of $H$ onto $M$* is the map $P : H \to H$ defined by

$$P(h_1 + h_2) = h_1, \qquad h_1 \in M, h_2 \in M^\perp.$$

It is straightforward to check that $P$ is linear, $\|P\| \leq 1$ ($\|P\| = 1$ if and only if $M$ is nonzero), $P^2 = P$, and $\ker P = M^\perp$ and $P(H) = M$.[7] Rather than specifying a closed subspace of $H$ and talking about the orthogonal projection onto $M$, we can talk about an orthogonal projection in $H$, which is the orthogonal projection onto its image.

Bessel's inequality[8] states that if $\{e_n : n \in \mathbb{N}\}$ is an orthonormal set in a Hilbert space $H$ and $h \in H$, then

$$\sum_{n=1}^{\infty} |\langle h, e_n\rangle|^2 \leq \|h\|^2. \tag{1}$$

**Theorem 8.** *If $H$ is a Hilbert space, $\mathscr{E}$ is an orthonormal set in $H$, and $h \in H$, then there are only countably many $e \in \mathscr{E}$ such that $\langle h, e\rangle \neq 0$.*

*Proof.* Let

$$\mathscr{E}_n = \left\{ e \in \mathscr{E} : |\langle h, e\rangle| \geq \frac{1}{n} \right\}.$$

If $\mathscr{E}_n$ were infinite, let $\{e_j : j \in \mathbb{N}\}$ be a subset of it, and this gives us a contradiction by (1). Therefore each $\mathscr{E}_n$ is finite. But if $\langle h, e\rangle \neq 0$ then there is

[6] John B. Conway, *A Course in Functional Analysis*, second ed., p. 9, Theorem 2.6.
[7] John B. Conway, *A Course in Functional Analysis*, second ed., p. 10, Theorem 2.7.
[8] John B. Conway, *A Course in Functional Analysis*, second ed., p. 15, Theorem 4.8.

some $n$ such that $|\langle h, e\rangle| \geq \frac{1}{n}$, so

$$\mathscr{E} = \bigcup_{n=1}^{\infty} \mathscr{E}_n.$$

Therefore $\mathscr{E}$ is countable. $\square$

Bessel's inequality makes sense for an orthonormal set of any cardinality in a Hilbert space, rather than just for a countable orthonormal set.

**Theorem 9** (Bessel's inequality)**.** *If $H$ is a Hilbert space, $\mathscr{E}$ is an orthonormal set in $H$, and $h \in H$, then*

$$\sum_{e \in \mathscr{E}} |\langle h, e\rangle|^2 \leq \|h\|^2 .$$

*Proof.* By Theorem 8, there are only countably many $e \in \mathscr{E}$ such that $\langle h, e\rangle \neq 0$; let them be $\{e_n : n \in \mathbb{N}\}$. $\{e_n : n \in \mathbb{N}\}$ is an orthonormal set, so by (1) we have

$$\sum_{n=1}^{\infty} |\langle h, e_n\rangle|^2 \leq \|h\|^2 .$$

Theorem 4 states that if $X$ is a Banach space, $x_n \in X, n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then the unordered sum $\sum_{n\in\mathbb{N}} x_n$ converges. Thus, with $X = \mathbb{C}$ and $x_n = |\langle h, e_n\rangle|^2$, the unordered sum $\sum_{n\in\mathbb{N}} |\langle h, e_n\rangle|^2$ converges, say to $S$. Because $\sum_{n\in\mathbb{N}} |\langle h, e_n\rangle|^2$ converges to $S$, by Theorem 3 the series $\sum_{n=1}^{\infty} |\langle h, e_n\rangle|^2$ converges to $S$. But we already know that this series is $\leq \|h\|^2$, so

$$\sum_{n\in\mathbb{N}} |\langle h, e_n\rangle|^2 \leq \|h\|^2 .$$

By Theorem 1, the unordered sum $\sum_{e\in\mathscr{E}} |\langle h, e\rangle|^2$ converges if and only if the unordered sum $\sum_{n\in\mathbb{N}} |\langle h, e_n\rangle|^2$ converges, and if they converge they have the same value. Therefore, the unordered sum $\sum_{e\in\mathscr{E}} |\langle h, e\rangle|^2$ indeed converges, and it is $\leq \|h\|^2$. $\square$

## 4 Convergence of unordered sums in the strong operator topology

Let $H$ be a Hilbert space and let $\mathscr{B}(H)$ be the set of bounded linear maps $H \to H$. It is straightforward to check that $\mathscr{B}(H)$ is a normed space with the operator norm $\|T\| = \sup_{\|h\|\leq 1} \|Th\|$. (In fact it is a Banach space, actually a Banach algebra, actually a $C^*$-algebra; each of these statements implies the previous one.) The *strong operator topology* on $\mathscr{B}(H)$ can be characterized in the following way: a net $f : I \to \mathscr{B}(H)$ converges to $T \in \mathscr{B}(H)$ in the strong operator topology if for all $h \in H$ the net $f(i)h$ converges to $Th$ in $H$.[9]

If $I$ is a set, $\mathscr{F}$ is the set of all finite subsets of $I$, and $f : I \to \mathscr{B}(H)$ is a function, define $S : \mathscr{F} \to \mathscr{B}(H)$ by

$$S(F) = \sum_{i \in I} f(i) \in \mathscr{B}(H).$$

$S$ is a net in $\mathscr{B}(H)$, and if the net converges to $T \in \mathscr{B}(H)$ in the strong operator topology we say that the unordered sum $\sum_{i \in I} f(i)$ *converges strongly* to $T$. To say that the net $S$ converges to $T$ in the strong operator topology is to say that if $h \in H$ then $\sum_{i \in I} f(i)h$ converges to $Th$ in $H$.

If $f, g \in H$, we define $f \otimes g : H \to H$ by

$$f \otimes g(h) = \langle h, g \rangle f.$$

It is apparent that $f \otimes g$ is linear, and

$$\|f \otimes g(h)\| = \|\langle h, g \rangle f\| = |\langle h, g \rangle| \, \|f\| \leq \|h\| \, \|g\| \, \|f\|,$$

so $\|f \otimes g\| \leq \|f\| \, \|g\|$, giving $f \otimes g \in \mathscr{B}(H)$. Additionally,

$$\langle f \otimes g(h_1), h_2 \rangle = \langle \langle h_1, g \rangle f, h_2 \rangle = \langle h_1, g \rangle \langle f, h_2 \rangle = \langle h_1, \langle h_2, f \rangle g \rangle = \langle h_1, g \otimes f(h_2) \rangle,$$

showing that $(f \otimes g)^* = g \otimes f$.

**Theorem 10.** *If $H$ is a Hilbert space, $\mathscr{E}$ is an orthonormal set in $H$, and $P$ is the orthogonal projection onto $\bigvee \mathscr{E}$, then $\sum_{e \in \mathscr{E}} e \otimes e$ converges strongly to $P$.*

*Proof.* Let $h \in H$. By Theorem 8 there are only countably many $e \in \mathscr{E}$ such that $\langle h, e \rangle \neq 0$, and we denote these by $\{e_n : n \in \mathbb{N}\}$. By Bessel's inequality,

$$\sum_{e \in \mathscr{E}} |\langle h, e \rangle|^2 = \sum_{n \in \mathbb{N}} |\langle h, e_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle h, e_n \rangle|^2 \leq \|h\|^2 . \tag{2}$$

Let $\mathscr{F}$ be the set of all finite subsets of $\mathbb{N}$ and for $F \in \mathscr{F}$ let

$$S(F) = \sum_{n \in F} \langle h, e_n \rangle e_n \in H.$$

If $\epsilon > 0$, then by (2) there is some $N$ such that $\sum_{n=N+1}^{\infty} |\langle h, e_n \rangle|^2 < \epsilon^2$. If $F_\epsilon = \{1, \ldots, N\}$ and $F, G \in \mathscr{F}$ both contain $F_\epsilon$, then, because the $e_n$ are

[9]For the strong operator topology see John B. Conway, *A Course in Functional Analysis*, second ed., p. 256.

orthonormal,

$$\begin{aligned}
\|S(F)-S(G)\|^2 &= \left\|\sum_{n\in F}\langle h,e_n\rangle e_n - \sum_{n\in G}\langle h,e_n\rangle e_n\right\|^2 \\
&= \sum_{n\in(F\cup G)\setminus(F\cap G)} \|\langle h,e_n\rangle e_n\|^2 \\
&= \sum_{n\in(F\cup G)\setminus(F\cap G)} |\langle h,e_n\rangle|^2 \\
&\leq \sum_{n=N+1}^{\infty} |\langle h,e_n\rangle|^2 \\
&< \epsilon^2.
\end{aligned}$$

Therefore, if $F, G \in \mathscr{F}$ both contain $F_\epsilon$ then $\|S(F)-S(G)\| < \epsilon$. This means that $S$ is a Cauchy net, and hence, by Theorem 6, has a limit $v \in H$. That is, the unordered sum $\sum_{n\in\mathbb{N}}\langle h,e_n\rangle e_n$ converges to $v$.

As the unordered sum $\sum_{n\in\mathbb{N}}\langle h,e_n\rangle e_n$ converges to $v$ we have

$$\lim_{N\to\infty}\sum_{n=1}^{N}\langle h,e_n\rangle e_n = v.$$

If $m \in \mathbb{N}$ then it follows that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\langle h,e_n\rangle\langle e_n,e_m\rangle = \langle v,e_m\rangle,$$

which is

$$\langle h,e_m\rangle = \langle v,e_m\rangle.$$

Let $Q$ be the orthogonal projection onto $\bigvee_{n\in\mathbb{N}}\{e_n\}$. On the one hand, because $\langle h,e\rangle = 0$ for $e \notin \{e_n : n \in \mathbb{N}\}$, we check that $Ph = Qh$. On the other hand, we check that $Qh = v$. Therefore, $v = Ph$, i.e.

$$\sum_{e\in\mathscr{E}} e\otimes e(h) = \sum_{e\in\mathscr{E}}\langle h,e\rangle e = \sum_{n\in\mathbb{N}}\langle h,e_n\rangle e_n = Ph,$$

showing that the unordered sum $\sum_{e\in\mathscr{E}} e\otimes e$ converges strongly to $P$. $\square$

In particular, if $\mathscr{E}$ is an orthonormal basis for $H$, then $\sum_{e\in\mathscr{E}} e\otimes e$ converges strongly to $\mathrm{id}_H$.