# The $C^\infty$ Urysohn lemma

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Define $\eta : \mathbb{R} \to \mathbb{R}$ by

$$\eta(t) = e^{-1/t} 1_{(0,\infty)}(t).$$

It is a fact that $\eta$ is $C^\infty$. This is proved by showing that for each $k \geq 1$ there is a polynomial $P_k$ of degree $2k$ such that $\eta^{(k)}(t) = P_k(t^{-1})e^{-1/t}$ for $t > 0$, and that $\eta^{(k)}(0) = 0$, which together imply that $\eta \in C^k$.

Define $\psi : \mathbb{R}^d \to \mathbb{R}$ by

$$\psi(x) = \eta(1 - |x|^2) = \begin{cases} e^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

Because $x \mapsto 1 - |x|^2$ is $C^\infty : \mathbb{R}^d \to \mathbb{R}$, the chain rule tells us that $\psi$ is $C^\infty$.

For a function $\phi$ on $\mathbb{R}^d$ and for $t > 0$, we define

$$\phi_t(x) = t^{-d}\phi(t^{-1}x).$$

We now construct bump functions.[1]

**Theorem 1** ($C^\infty$ Urysohn lemma)**.** *If $K$ is a compact subset of $\mathbb{R}^d$ and $U$ is an open set containing $K$, then there exists $\phi \in C^\infty(\mathbb{R}^d)$ with $0 \leq \phi \leq 1$, $\phi = 1$ on $K$, and $\operatorname{supp} \phi \subset U$. Moreover, if $K$ is invariant under $SO(d)$ then the function $\phi$ constructed here is radial.*

*Proof.* Let

$$\delta = d(K, U^c),$$

which is positive because $K$ is compact and $U^c$ is closed. Let

$$V = \left\{ x \in \mathbb{R}^d : d(x, K) < \frac{\delta}{3} \right\} = K + B_{\delta/3},$$

and define $f$ on $\mathbb{R}^d$ by

$$f = \left( \int_{\mathbb{R}^d} \psi(x) dx \right)^{-1} \psi_{\delta/3},$$

[1] The following construction of a bump function follows Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, second ed., p. 245, Lemma 8.18.

whose support is

$$\operatorname{supp} f = \operatorname{supp} \psi_{\delta/3} = \overline{B_{\delta/3}}.$$

Finally define $\phi$ on $\mathbb{R}^d$ by

$$\phi = 1_V * f.$$

Because $V$ is bounded and $f$ is $C^\infty$, the function $\phi$ is $C^\infty$. The support of $\phi$ is

$$\operatorname{supp} \phi = \operatorname{supp} (1_V * f) \subset \overline{\operatorname{supp} 1_V + \operatorname{supp} f} = \overline{V + \overline{B_{\delta/3}}} = K + \overline{B_{2\delta/3}} \subset U.$$

Because $1_V$ and $f$ are nonnegative, so is their convolution $\phi$. For any $x$,

$$\phi(x) = \int_{\mathbb{R}^d} 1_V(x-y) f(y) dy \le \int_{\mathbb{R}^d} f(y) dy = 1,$$

so $0 \le \phi \le 1$. For $x \in K$, if $y \in V^c$ then $|x - y| \ge \delta/3$. But $f(u) = 0$ for $|u| \ge \delta/3$, so in this case $f(x-y) = 0$. This implies that for $x \in K$ the functions $y \mapsto 1_V(y) f(x-y)$ and $y \mapsto f(x-y)$ are equal, hence

$$\phi(x) = \int_{\mathbb{R}^d} 1_V(y) f(x-y) dy = \int_{\mathbb{R}^d} f(x-y) dy = \int_{\mathbb{R}^d} f(y) dy = 1.$$

This shows that $\phi = 1$ on $K$, verifying all the assertions made about $\phi$.

The function $\psi$ is radial and so $f$ is too. If $V$ is invariant under $SO(d)$, then the indicator function $1_V$ is radial. Thus, if $K$ is invariant under $SO(d)$ then $1_V$ is radial, and the convolution of two radial functions is also radial, which means that $\phi$ is radial in this case. $\square$

For example, take $d = 1$, take $K$ to be the closed ball of radius 1, and take $U$ to be the open ball of radius 2. Then $\delta = d(K, U^c) = 1$ and $V = B_{4/3}$. In Figure 1 we plot the bump function $\phi$ constructed in the above theorem.

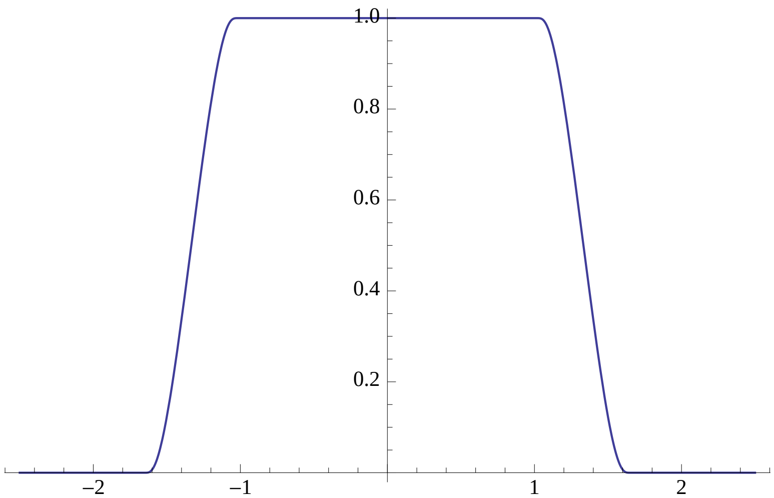


Figure 1: The bump function $\phi$, for $d = 1$, $K = [-1, 1]$, $U = (-2, 2)$; $\delta = 1$ and $V = (-4/3, 4/3)$