Valued fields

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1 Absolute values

Let F be a field. An **absolute value on** F is a function $|\cdot| : F \to \mathbb{R}_{\geq 0}$ such that (i) $|x| = 0$ implies $x = 0$, (ii) $|x+y| \leq |x|+|y|$, and (iii) $|xy| = |x||y|$. The pair F and $|\cdot|$ is called a **valued field**. An absolute value is called **nonarchimedean** if

$$|x + y| \leq \max(|x|, |y|),$$

and **archimedean** otherwise. Because $|xy| = |x||y|$, $x \mapsto |x|$ is a group homomorphism $F^* \to \mathbb{R}_{>0}$. The **trivial absolute value** is $|x| = 0$ for $x = 0$ and $|x| = 1$ for $x \neq 1$, which is nonarchimedean.

The **value group** of F is the image of F^* under $x \mapsto |x|$; it is a subgroup of $\mathbb{R}_{>0}$. If the subspace topology on the value group inherited from $\mathbb{R}_{>0}$ is discrete, then the absolute value is called **discrete**. For example, it is a fact that the value group of $(\mathbb{Q}_p, |\cdot|_p)$ is $\{p^n : n \in \mathbb{Z}\}$, which is discrete.

If $|\cdot|$ is an absolute value on a field F, let $d(x, y) = |x - y|$. This is a metric on F, and with the topology induced by d, F is a topological field.[1] We call the valued field F **complete** if d is a complete metric.

If $(F, |\cdot|_F)$ and $(K, |\cdot|_K)$ are valued fields, a **homomorphism of valued fields** from F to K is a field homomorphism $\phi : F \to K$ such that $|\phi(x)|_K = |x|_F$ for all $x \in F$. If ϕ is onto then ϕ is called an **isomorphism of valued fields**.

If F is a field with a nontrivial absolute value $|\cdot|_F$, then there is a complete valued field $(K, |\cdot|_K)$ and a homomorphism of valued fields $\iota : F \to K$ such that $\iota(F)$ is dense in K.[2] We call $(K, |\cdot|_K)$ a **completion of** F, and if F is nonarchimedean then it has a nonarchimedean completion. Completions of valued fields have the following **universal property**: if $\iota : (F, |\cdot|_F) \to (K, |\cdot|_K)$ is a completion of the valued field F, if $(L, |\cdot|_L)$ is a complete valued field, and if $\phi : F \to L$ is a homomorphism of valued fields, then there is a unique homomorphism of valued fields $\Phi : K \to L$ such that $\phi = \Phi \circ \iota$.[3] It is often

[1] Anthony W. Knapp, *Advanced Algebra*, p. 334, Proposition 6.13; the proof is straightforward.

[2] Anthony W. Knapp, *Advanced Algebra*, p. 342, Theorem 6.24; W. H. Schikhof, *Ultrametric calculus: An introduction to p-adic analysis*, p. 15, Theorem 6.3.

[3] Anthony W. Knapp, *Advanced Algebra*, p. 343, Theorem 6.25.

permissible to talk **the completion** $(K, |\cdot|_K)$ of a valued field $(F, |\cdot|_F)$ and $F \subset K$ where the restriction of $|\cdot|_K$ to F is $|\cdot|_F$, rather than $\iota : F \to K$; for some arguments it is necessary to speak about $\iota : F \to K$ rather than $F \subset K$.

2 Nonarchimedean valued fields

Let F be a field with a nontrivial nonarchimedean absolute value $|\cdot|_F$. For $a \in F$ and $r > 0$ let

$$B_{\leq r}(a) = \{x \in F : |x-a|_F \leq r\}, \qquad B_{<r}(a) = \{x \in F : |x-a|_F < r\}.$$

We now prove that $B_{\leq 1}(0)$ is a local ring whose unique maximal ideal is $B_{<1}(0)$.[4] (A commutative ring R is a **local ring** if it has a unique maximal ideal.)

Lemma 1. *$B_{\leq 1}(0)$ is a local ring and $B_{<1}(0)$ is the unique maximal ideal in $B_{\leq 1}(0)$.*

Proof. If $x, y \in B_{\leq 1}(0)$ then $|x+y|_F \leq \max(|x|_F, |y|_F) \leq 1$ so $x + y \in B_{\leq 1}(0)$. $|-x|_F = |x|_F \leq 1$ so $-x \in B_{\leq 1}(0)$. $|xy|_F = |x|_F|y|_F \leq 1$ so $xy \in B_{\leq 1}(0)$. $|1|_F = 1$ so $1 \in B_{\leq 1}(0)$. Therefore $B_{\leq 1}(0)$ is a subring of K.

For $x, y \in B_{<1}(0)$, $|x+y|_F \leq \max(|x|_F, |y|_F) < 1$ so $x + y \in B_{<1}(0)$. For $x \in B_{<1}(0)$ and $y \in B_{\leq 1}(0)$, $|xy|_F = |x|_F|y|_F < 1$ so $xy \in B_{<1}(0)$ and therefore $B_{<1}(0)$ is an ideal in the ring $B_{\leq 1}(0)$. Now, if $|x|_F = 1$ then $x^{-1} \in F$ satisfies $|x^{-1}|_F = 1$. Therefore, $B_{<1}(0)$ is the set of elements in $B_{\leq 1}(0)$ which do not have an inverse in $B_{\leq 1}(0)$. Generally, if R is a commutative ring and the set M of noninvertible elements is an ideal, then R is a local ring with unique maximal ideal M. □

The **residue class field of** F is the field

$$B_{\leq 1}(0)/B_{<1}(0).$$

It can be proved that a complete nonarchimedean field is locally compact if and only if its residue class field is finite and the absolute value is discrete.[5]

3 Algebraic closures

Let $(F, |\cdot|_F)$ be a nonarchimedean valued field and let K be a field containing F. It can be proved that there is a nonarchimedean absolute value on K whose restriction to F is equal to $|\cdot|_F$.[6] Furthermore, if F is complete and K is algebraic over F then this absolute value on K is unique.[7]

[4] W. H. Schikhof, *Ultrametric calculus: An introduction to p-adic analysis*, p. 25, Proposition 11.1.

[5] W. H. Schikhof, *Ultrametric calculus: An introduction to p-adic analysis*, p. 29, Corollary 12.2.

[6] W. H. Schikhof, *Ultrametric calculus: An introduction to p-adic analysis*, p. 34, Theorem 14.1.

[7] W. H. Schikhof, *Ultrametric calculus: An introduction to p-adic analysis*, p. 35, Theorem 14.2.

If $(F, |\cdot|_F)$ is a complete nonarchimedean valued field and K is an algebraic closure of F, then by the above there is a unique nonarchimedean absolute value $|\cdot|_K$ on K such that $|x|_K = |x|_F$ for $x \in F$. If $(K, |\cdot|_K)$ is not complete, then we have stated earlier that it has a completion $(L, |\cdot|_L)$. It can in fact be proved that the field L is algebraically closed.[8]

Let $\mathbb{Q}_p^a$ be an algebraic closure of $\mathbb{Q}_p$. Then there is a unique nonarchimedean absolute value on $\mathbb{Q}_p^a$ whose restriction to $\mathbb{Q}_p$ is equal to $|\cdot|_p$. One proves that the valued field $\mathbb{Q}_p^a$ is not complete.[9] Let $\mathbb{C}_p$ be the completion of the valued field $\mathbb{Q}_p^a$, which by what we have said is an algebraically closed nonarchimedean valued field. $\mathbb{Q}_p \subset \mathbb{C}_p$, and $|x|_{\mathbb{C}_p} = |x|_p$ for $x \in \mathbb{Q}_p$. One further proves that the residue class field of $\mathbb{C}_p$ is the algebraic closure of $\mathbb{F}_p$, and hence $\mathbb{C}_p$ is not locally compact. The value group of $\mathbb{C}_p$ is $\{p^r : r \in \mathbb{Q}\}$. Finally, $\mathbb{C}_p$ is a separable metric space.

It turns out to be fruitful to work with functions $\mathbb{Z}_p \to \mathbb{C}_p$, and because $\mathbb{C}_p$ is an occult object it is useful to become familiar with it before working out this machinery.

[8] W. H. Schikhof, *Ultrametric calculus: An introduction to p-adic analysis*, p. 45, Theorem 17.1.

[9] W. H. Schikhof, *Ultrametric calculus: An introduction to p-adic analysis*, p. 43, Theorem 16.6.