Vinogradov's estimate for exponential sums over primes

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1 Introduction

For $x \in \mathbb{R}$, let $[x]$ be the greatest integer $\leq x$, let $R(x) = x - [x]$, and let

$$\|x\| = \min\{R(x), 1 - R(x)\} = \min_{m \in \mathbb{Z}} |x - m|.$$

In this note I work through Chapters 24 and 25 of Harold Davenport, *Multiplicative Number Theory*, third ed.[1]

We end up proving that there is some constant C such that if $\alpha \in \mathbb{R}$ and $\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^2}$, with a positive and $\gcd(a, q) = 1$, then for any $N \geq 2$,

2 The von Mangoldt function

Let Λ be the von Mangoldt function: $\Lambda(n) = \log p$ if $n = p^\alpha$ for some prime p and $\alpha \geq 1$, and $\Lambda(n) = 0$ otherwise. For example, $\Lambda(4) = \log 2$, $\Lambda(11) = \log 11$, and $\Lambda(12) = 0$. This satisfies

$$\log n = \sum_{d|n} \Lambda(d),$$

and so by the Möbius inversion formula,

$$\Lambda(n) = \sum_{d|n} \mu(n/d) \log d.$$

[1]Many of the manipulations of sums in these chapters are hard to follow, and I greatly expand on the calculations in Davenport. The organization of the proof in Davenport seems to be due to Vaughan. I have also used lecture notes by Andreas Strömbergsson, http://www2.math.uu.se/~astrombe/analtalt08/www_notes.pdf, pp. 245–257. Another set of notes, which I have not used, are http://jonismathnotes.blogspot.ca/2014/11/prime-exponential-sums-and-vaughans.html

For $n > 1$ it is a fact that[2]

$$\sum_{d|n} \mu(d) = 0.$$

Write $\psi(x) = \sum_{n \le x} \Lambda(n)$. It is a fact that $\psi(x) = O(x)$.[3] (The **prime number theorem** states $\psi(x) \sim x$.)

The derivative of the Riemann zeta function is

$$\zeta'(s) = -\sum_{n=1}^{\infty} n^{-s} \log n, \qquad \mathrm{Re}\, s > 1.$$

The Euler product for the Riemann zeta function is

$$\zeta(s) = \sum_{m=1}^{\infty} m^{-s} = \prod_p (1 - p^{-s})^{-1}, \qquad \mathrm{Re}\, s > 1.$$

Then

$$-\log \zeta(s) = \sum_p \log(1 - p^{-s}),$$

so, for $\mathrm{Re}\, s > 1$,

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= \sum_p \frac{p^{-s} \log p}{1 - p^{-s}} \\ &= \sum_p \log p \sum_{l=1}^{\infty} p^{-ls} \\ &= \sum_p \sum_{l=1}^{\infty} \Lambda(p^l)(p^l)^{-s} \\ &= \sum_{n=1}^{\infty} \Lambda(n) n^{-s}. \end{aligned}$$

Let $U, V \ge 2$, $UV \le N$, and write

$$F_U(s) = \sum_{j \le U} \Lambda(j) j^{-s}, \qquad G_V(s) = \sum_{k \le V} \mu(k) k^{-s}.$$

For $\mathrm{Re}\, s > 1$,

$$\begin{aligned} & -\frac{\zeta'(s)}{\zeta(s)} \\ =& F_U(s) - \zeta(s) F_U(s) G_V(s) - \zeta'(s) G_V(s) + \left(-\frac{\zeta'(s)}{\zeta(s)} - F_U(s) \right) (1 - \zeta(s) G_V(s)). \end{aligned}$$

[2] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, fifth ed., p. 235, Theorem 263.

[3] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, fifth ed., p. 341, Theorem 414.

First,[4]

$$F_U(s) = \sum_{n=1}^{\infty} a_1(n) n^{-s}$$

for

$$a_1(n) = \begin{cases} \Lambda(n) & n \leq U \\ a_1(n) = 0 & n > U. \end{cases}$$

Second,

$$-\zeta(s) F_U(s) G_V(s) = \sum_{n=1}^{\infty} a_2(n) n^{-s}$$

for

$$a_2(n) = -\sum_{mjk=n, m\geq 1, j\leq U, k\leq V} 1 \cdot \Lambda(j) \cdot \mu(k) = -\sum_{d|n} \sum_{djk=n, j\leq U, k\leq V} \Lambda(j)\mu(k).$$

Third,

$$-\zeta'(s) G_V(s) = \sum_{n=1}^{\infty} a_3(n) n^{-s}$$

for

$$a_3(n) = \sum_{mk=n, m\geq 1, k\leq V} \log(m) \cdot \mu(k) = \sum_{d|n} \log d \sum_{dk=n, k\leq V} \mu(k).$$

Fourth,

$$-\frac{\zeta'(s)}{\zeta(s)} - F_U(s) = \sum_{m>U} \Lambda(m) m^{-s};$$

and

$$\zeta(s) G_V(s) = \sum_{n=1}^{\infty} \left(\sum_{mk=n, m\geq 1, k\leq V} 1 \cdot \mu(k) \right) n^{-s} = \sum_{n=1}^{\infty} \left(\sum_{d|n, d\leq V} \mu(d) \right) n^{-s}$$

whence

$$1 - \zeta(s) G_V(s) = -\sum_{h=2}^{\infty} \left(\sum_{d|h, d\leq V} \mu(d) \right) h^{-s};$$

thus

$$\left(-\frac{\zeta'(s)}{\zeta(s)} - F_U(s) \right) (1 - \zeta(s) G_V(s)) = \sum_{n=1}^{\infty} a_4(n) n^{-s}$$

for

$$a_4(n) = -\sum_{mh=n, m>U, h>1} \Lambda(m) \left(\sum_{d|h, d\leq V} \mu(d) \right).$$

We have

$$\Lambda(n) = a_1(n) + a_2(n) + a_3(n) + a_4(n).$$

[4] Harold Davenport, *Multiplicative Number Theory*, third ed., p. 138, Chapter 24.

3 Sums involving the von Mangoldt function

Let f be an arithmetical function with $|f| \le 1$ and write

$$S_i = \sum_{n \le N} f(n) a_i(n),$$

for which

$$\sum_{n \le N} f(n)\Lambda(n) = S_1 + S_2 + S_3 + S_4.$$

$$\begin{aligned}
S_1 &= \sum_{n \le U} f(n)\Lambda(n) \\
S_2 &= -\sum_{n \le N} f(n) \sum_{d|n} \sum_{djk=n, j \le U, k \le V} \Lambda(j)\mu(k) \\
S_3 &= \sum_{n \le N} f(n) \sum_{d|n} \log d \sum_{dk=n, k \le V} \mu(k) \\
S_4 &= -\sum_{n \le N} f(n) \sum_{mh=n, m>U, h>1} \Lambda(m) \sum_{d|h, d \le V} \mu(d).
\end{aligned}$$

Lemma 1. $|S_1| = O(U)$.

Proof. As $|f| \le 1$,

$$|S_1| \le \sum_{n \le U} \Lambda(n) = \psi(U) = O(U).$$

□

Lemma 2.

$$|S_2| \le \log UV \cdot \sum_{h \le UV} \left| \sum_{r \le N/h} f(rh) \right|.$$

Proof.

$$\begin{aligned}
S_2 &= -\sum_{n \le N} f(n) \sum_{d|n} \sum_{djk=n, j \le U, k \le V} \Lambda(j)\mu(k) \\
&= -\sum_{h \le UV} \left(\sum_{jk=h, j \le U, k \le V} \Lambda(j)\mu(k) \right) \sum_{r \le N/h} f(rh).
\end{aligned}$$

For $h \le UV$, $\sum_{j|h} \Lambda(j) = \log h \le \log UV$, so

$$|S_2| \le \log UV \cdot \sum_{h \le UV} \left| \sum_{r \le N/h} f(rh) \right|.$$

□

Lemma 3.

$$|S_3| \le \log N \cdot \sum_{k\le V} \max_{1\le w\le N/k} \left| \sum_{w\le h\le N/k} f(kh) \right|.$$

Proof.

$$\begin{aligned} S_3 &= \sum_{n\le N} f(n) \sum_{d|n} \log d \sum_{dk=n, k\le V} \mu(k) \\ &= \sum_{k\le V} \mu(k) \sum_{h\le N/k} f(kh) \log h \\ &= \sum_{k\le V} \mu(k) \sum_{h\le N/k} f(kh) \int_1^h \frac{dw}{w} \\ &= \sum_{k\le V} \mu(k) \int_1^{N/k} \sum_{w\le h\le N/k} f(kh) \frac{dw}{w} \\ &= \int_1^N \sum_{k\le V} \mu(k) \sum_{w\le h\le N/k} f(kh) \frac{dw}{w}. \end{aligned}$$

Then

$$\begin{aligned} |S_3| &\le \max_{1\le w\le N} \left| \sum_{k\le V} \mu(k) \sum_{w\le h\le N/k} f(kh) \right| \cdot \int_1^N \frac{dw}{w} \\ &\le \log N \cdot \sum_{k\le V} \max_{1\le w\le N/k} \left| \sum_{w\le h\le N/k} f(kh) \right|. \end{aligned}$$

□

Lemma 4.

$$|S_4| \ll N^{1/2} (\log N)^3 \max_{U\le M\le N/V} \Delta,$$

for

$$\Delta = \max_{V<j\le N/M} \left(\sum_{V<k\le N/M} \left| \sum_{M\le m\le 2M, m\le N/j, m\le N/k} f(mj)\overline{f(mk)} \right| \right)^{1/2}.$$

Proof. For $b_m, c_k \in \mathbb{C}$, using the Cauchy-Schwarz inequality,

$$\begin{aligned} &\left| \sum_{M<m\le 2M} b_m \sum_{V<k\le N/m} c_k f(mk) \right| \\ &\le \left(\sum_{M\le m\le 2M} |b_m|^2 \right)^{1/2} \left(\sum_{M\le m\le 2M} \left| \sum_{V<k\le N/m} c_k f(mk) \right|^2 \right)^{1/2}, \end{aligned}$$

and

$$
\begin{aligned}
&\sum_{M\le m\le 2M}\left|\sum_{V<k\le N/m} c_k f(mk)\right|^2\\
&=\sum_{M\le m\le 2M}\left(\sum_{V<j\le N/m} c_j f(mj)\right)\left(\sum_{V<k\le N/m}\overline{c_k f(mk)}\right)\\
&=\sum_{V<j\le N/M}\sum_{V<k\le N/M} c_j\overline{c_k}\sum_{M\le m\le 2M, m\le N/j, m\le N/k} f(mj)\overline{f(mk)},
\end{aligned}
$$

so, using $|c_j c_k|\le \frac{1}{2}|c_j|^2+\frac{1}{2}|c_k|^2$,

$$
\begin{aligned}
&\sum_{M\le m\le 2M}\left|\sum_{V<k\le N/m} c_k f(mk)\right|^2\\
&\le\sum_{V<j\le N/M}\sum_{V<k\le N/M}\left(\frac{1}{2}|c_j|^2+\frac{1}{2}|c_k|^2\right)\left|\sum_{M\le m\le 2M, m\le N/j, m\le N/k} f(mj)\overline{f(mk)}\right|\\
&=\sum_{V<j\le N/M}|c_j|^2\sum_{V<k\le N/M}\left|\sum_{M\le m\le 2M, m\le N/j, m\le N/k} f(mj)\overline{f(mk)}\right|\\
&\le\left(\sum_{V<j\le N/M}|c_j|^2\right)\max_{V<j\le N/M}\sum_{V<k\le N/M}\left|\sum_{M\le m\le 2M, m\le N/j, m\le N/k} f(mj)\overline{f(mk)}\right|.
\end{aligned}
$$

Therefore,

$$
\left|\sum_{M<m\le 2M} b_m\sum_{V<k\le N/m} c_k f(mk)\right|\le\Delta\left(\sum_{M\le m\le 2M}|b_m|^2\right)^{1/2}\left(\sum_{j\le N/M}|c_j|^2\right)^{1/2}
$$

for

$$
\Delta=\left(\max_{V<j\le N/M}\sum_{V<k\le N/M}\left|\sum_{M\le m\le 2M, m\le N/j, m\le N/k} f(mj)\overline{f(mk)}\right|\right)^{1/2}.
$$

Because $\sum_{d|h}\mu(d)=0$ for $h>1$, if $1<h\le V$ then $\sum_{d|h, d\le V}\mu(d)=0$.

Thus

$$\begin{aligned}
S_4 &= -\sum_{n\le N} f(n) \sum_{mh=n, m>U, h>1} \Lambda(m) \sum_{d|h, d\le V} \mu(d)\\
&= -\sum_{U<m<N/V} \Lambda(m) \sum_{V<k\le N/m} f(mk) \sum_{d|k, d\le V} \mu(d)\\
&= -\sum_{U<m\le N/V} \Lambda(m) \sum_{V<k\le N/m} f(mk) \sum_{d|k, d\le V} \mu(d)\\
&= -\sum_{M\in\{U,2U,4U,\ldots\}, M<N/V} \ \sum_{M<m\le \min(N/V,2M)} \Lambda(m) \sum_{V<k\le N/m} f(mk) \sum_{d|k, d\le V} \mu(d),
\end{aligned}$$

so

$$|S_4| \le \left(\log_2 \frac{N}{UV}\right) \max_{U\le M\le N/V} \left| \sum_{M<m\le \min(N/V,2M)} \Lambda(m) \sum_{V<k\le N/m} f(mk) \sum_{d|k, d\le V} \mu(d) \right|.$$

Define $b_m = \Lambda(m)$ for $m \le N/V$ and $b_m = 0$ for $m > N/V$, and $c_k = \sum_{d|k, d\le V} \mu(d)$. Using the above we get

$$\begin{aligned}
|S_4| &\ll (\log N) \max_{U\le M\le N/V} \left| \sum_{M<m\le 2M} b_m \sum_{V<k\le N/m} c_k f(mk) \right|\\
&\ll (\log N) \max_{U\le M\le N/V} \Delta \left(\sum_{M\le m\le 2M} |b_m|^2 \right)^{1/2} \left(\sum_{j\le N/M} |c_j|^2 \right)^{1/2}\\
&\ll (\log N) \max_{U\le M\le N/V} \Delta \left(\sum_{M\le m\le 2M} \Lambda(m)^2 \right)^{1/2} \left(\sum_{j\le N/M} d(k)^2 \right)^{1/2}
\end{aligned}$$

On the one hand,

$$\sum_{m\le y} \Lambda(m)^2 \le (\log y) \sum_{m\le y} \Lambda(m) = O(y \log y).$$

On the other hand, let h be the multiplicative arithmetic function such that for prime p and for nonnegative integer a, $h(p^a) = 2a + 1$. The divisor function satisfies $d(p^a) = a + 1$, and

$$\begin{aligned}
\sum_{d|p^a} h(d) &= \sum_{0\le b\le a} h(p^b) \\
&= \sum_{0\le b\le a} (2b+1) \\
&= a + 1 + 2 \sum_{0\le b\le a} b \\
&= a + 1 + 2\cdot\frac{a(a+1)}{2} \\
&= a + 1 + a^2 + a \\
&= a^2 + 2a + 1 \\
&= d(p^a)^2.
\end{aligned}$$

Hence, as $d \mapsto \frac{h(d)}{d}$ is multiplicative and nonnegative,

$$\begin{aligned}
\sum_{k\le y} d(k)^2 &= \sum_{k\le y}\sum_{d|k} h(d) \\
&= \sum_{d\le y} h(d) \sum_{kd\le y} 1 \\
&= \sum_{d\le y} h(d)\cdot [y/d] \\
&\le y \sum_{d\le y} \frac{h(d)}{d} \\
&\le y \prod_{p\le y}\sum_{a=0}^{\infty} h(p^a)p^{-a} \\
&= y \prod_{p\le y}\sum_{a=0}^{\infty} (2a+1)p^{-a}.
\end{aligned}$$

But, for $0 < x < 1$,

$$(1-x)^{-3} = \left(\sum_{a=0}^{\infty} x^a\right)^3 = \sum_{a=0}^{\infty} \frac{1}{2}(a+1)(a+2)x^a \ge \sum_{a=0}^{\infty}(2a+1)x^a,$$

so

$$\sum_{k\le y} d(k)^2 \le y\left(\sum_{p\le y}\left(1-p^{-1}\right)^{-1}\right)^3.$$

Merten's theorem[5] tells us

$$\prod_{p \le y} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log y},$$

where γ is Euler's constant, and using this,

$$\sum_{k \le y} d(k)^2 = O(y(\log y)^3).$$

We have therefore got for $U \le M \le N/V$,

$$\begin{aligned}
&\left(\sum_{M \le m \le 2M} \Lambda(m)^2\right)^{1/2} \left(\sum_{k \le N/M} d(k)^2\right)^{1/2} \\
\le &\left(\sum_{m \le 2M} \Lambda(m)^2\right)^{1/2} \left(\sum_{k \le N/M} d(k)^2\right)^{1/2} \\
\le &\left(O(M \log M)\right)^{1/2} \left(O(N/M(\log(N/M))^3\right)^{1/2} \\
= &O(N^{1/2}(\log N)^2).
\end{aligned}$$

What we now have is

$$|S_4| \ll (\log N) \cdot N^{1/2} (\log N)^2 \cdot \max_{U \le M \le N/V} \Delta,$$

proving the claim. $\square$

Putting together the estimates for S_1, S_2, S_3, S_4 gives, for $|f| \le 1$, and $U, V \ge 2$, $UV \le N$,

$$\begin{aligned}
\sum_{n \le N} f(n)\Lambda(n) \ll U &+ (\log N) \sum_{h \le UV} \left| \sum_{r \le N/h} f(rh) \right| \\
&+ (\log N) \sum_{k \le V} \max_{1 \le w \le N/k} \left| \sum_{w \le h \le N/k} f(kh) \right| \\
&+ N^{1/2} (\log N)^3 \max_{U \le M \le N/V} \Delta,
\end{aligned}$$

for

$$\Delta = \max_{V < j \le N/M} \left(\sum_{V < k \le N/M} \left| \sum_{M \le m \le 2M, m \le N/j, m \le N/k} f(mj)\overline{f(mk)} \right| \right)^{1/2}.$$

[5] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, fifth ed., p. 351, Theorem 429.

4 Exponential sums

For $\beta \in \mathbb{R}$, on the one hand

$$\left| \sum_{N_1 \le n \le N_2} e^{2\pi i \beta n} \right| \le N_2 - N_1 + 1.$$

On the other hand,

$$\sum_{N_1 \le n \le N_2} e^{2\pi i \beta n} = \sum_{0 \le n \le N_2 - N_1} e^{2\pi i \beta n} = \frac{1 - e^{2\pi i \beta (N_2 - N_1 + 1)}}{1 - e^{2\pi i \beta}}$$

and hence

$$\left| \sum_{N_1 \le n \le N_2} e^{2\pi i \beta n} \right| \le \frac{2}{|1 - e^{2\pi i \beta}|} = \frac{1}{|\sin \pi \beta|} \le \frac{1}{2\,\|\beta\|}.$$

Thus

$$\left| \sum_{N_1 \le n \le N_2} e^{2\pi i \beta n} \right| \ll \min \left\{ N_2 - N_1, \frac{1}{\|\beta\|} \right\}.$$

Let $\alpha \in \mathbb{R}$ and let $f(n) = e^{2\pi i \alpha n}$. Then

$$\sum_{h \le UV} \left| \sum_{r \le N/h} f(rh) \right| \le \sum_{h \le UV} \min \left\{ \frac{N}{h}, \frac{1}{\|h\alpha\|} \right\}$$

and

$$\begin{aligned} \sum_{k \le V} \max_{w} \left| \sum_{w \le h \le N/k} f(rt) \right| &\ll \sum_{k \le V} \max_{w} \min \left\{ \frac{N}{k} - w, \frac{1}{\|k\alpha\|} \right\} \\ &\ll \sum_{k \le V} \min \left\{ \frac{N}{k}, \frac{1}{\|k\alpha\|} \right\}. \end{aligned}$$

Let $S_N(\alpha) = \sum_{n \le N} \Lambda(n) f(n)$. By what we have worked out,

$$\begin{aligned} |S_N(\alpha)| \ll U &+ (\log N) \sum_{h \le UV} \min \left\{ \frac{N}{h}, \frac{1}{\|h\alpha\|} \right\} \\ &+ (\log N) \sum_{k \le V} \min \left\{ \frac{N}{k}, \frac{1}{\|k\alpha\|} \right\} \\ &+ N^{1/2} (\log N)^3 \max_{U \le M \le N/V} \Delta, \end{aligned}$$

for

$$\Delta = \max_{V<j\le N/M}\left(\sum_{V<k\le N/M}\left|\sum_{M\le m\le 2M, m\le N/j, m\le N/k} f(mj)\overline{f(mk)}\right|\right)^{1/2}.$$

We calculate

$$\sum_{M\le m\le 2M, m\le N/j, m\le N/k} f(mj)\overline{f(mk)} = \sum_{M\le m\le 2M, m\le N/j, m\le N/k} e^{2\pi i\alpha m(j-k)}$$

so

$$\left|\sum_{M\le m\le 2M, m\le N/j, m\le N/k} f(mj)\overline{f(mk)}\right| \ll \min\left\{M, \frac{1}{\|(j-k)\alpha\|}\right\}.$$

We now have

$$\begin{aligned}|S_N(\alpha)| &\ll U + (\log N)\sum_{h\le UV}\min\left\{\frac{N}{h}, \frac{1}{\|h\alpha\|}\right\}\\ &+ (\log N)\sum_{k\le V}\min\left\{\frac{N}{k}, \frac{1}{\|k\alpha\|}\right\}\\ &+ N^{1/2}(\log N)^3\max_{U\le M\le N/V}\max_{V<j\le N/M}\left(\sum_{V<k\le N/M}\min\left\{M, \frac{1}{\|(k-j)\alpha\|}\right\}\right)^{1/2}.\end{aligned}$$

But for $V < j \le N/M$, a fortiori $0 \le j \le N/M$, whence

$$\begin{aligned}\sum_{V<k\le N/M}\min\left\{M, \frac{1}{\|(k-j)\alpha\|}\right\} &\le \sum_{0\le k\le N/M}\min\left\{M, \frac{1}{\|(k-j)\alpha\|}\right\}\\ &\le \sum_{|m|\le N/M}\min\left\{M, \frac{1}{\|m\alpha\|}\right\}\\ &= M + 2\sum_{1\le m\le N/M}\min\left\{M, \frac{1}{\|m\alpha\|}\right\}\\ &\ll M + \sum_{1\le m\le N/M}\min\left\{\frac{N}{m}, \frac{1}{\|m\alpha\|}\right\}.\end{aligned}$$

Summarizing, we have the following.

Theorem 5. *For $\alpha\in\mathbb{R}$ and $U, V \ge 2, UV \le N$,*

$$\begin{aligned}|S_N(\alpha)| &\ll U + (\log N)\sum_{h\le UV}\min\left\{\frac{N}{h}, \frac{1}{\|h\alpha\|}\right\}\\ &+ N^{1/2}(\log N)^3\max_{U\le M\le N/V}\left(M + \sum_{1\le m\le N/M}\min\left\{\frac{N}{m}, \frac{1}{\|m\alpha\|}\right\}\right)^{1/2}.\end{aligned}$$

5 Diophantine approximation

Theorem 6. *There is some C such that for all $0 < \alpha < 1$, if $\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^2}$,* $\gcd(a,q) = 1$, *and $T \geq 1$, then*

$$\sum_{t \leq T} \min\left\{\frac{N}{t}, \frac{1}{\|t\alpha\|}\right\} \leq C\left(\frac{N}{q} + T + q\right)\log(2qT).$$

Proof. Write $\beta = \alpha - \frac{a}{q}$. Then

$$\begin{aligned}\sum_{t \leq T} \min\left\{\frac{N}{t}, \frac{1}{\|t\alpha\|}\right\} &\leq \sum_{0 \leq h \leq T/q} \sum_{1 \leq r \leq q} \min\left\{\frac{N}{hq+r}, \frac{1}{\|hq\alpha + r\alpha\|}\right\} \\ &= \sum_{0 \leq h \leq T/q} \sum_{1 \leq r \leq q} \min\left\{\frac{N}{hq+r}, \frac{1}{\left\|\frac{ra}{q} + hq\beta + r\beta\right\|}\right\}.\end{aligned}$$

If $h = 0$ and $1 \leq r \leq \frac{q}{2}$, then, using $\|x - y\| \geq \|x\| - \|y\|$ and $|\beta| \leq \frac{1}{q^2}$,

$$\frac{1}{\left\|\frac{ra}{q} + hq\beta + r\beta\right\|} = \frac{1}{\left\|\frac{ra}{q} + r\beta\right\|} \leq \frac{1}{\left\|\frac{ra}{q}\right\| - \|r\beta\|} \leq \frac{1}{\left\|\frac{ra}{q}\right\| - \frac{1}{2q}},$$

and, as $\gcd(a,q) = 1$,

$$\begin{aligned}\sum_{1 \leq r \leq \frac{q}{2}} \frac{1}{\left\|\frac{ra}{q}\right\| - \frac{1}{2q}} &\leq \sum_{1 \leq m < q} \frac{1}{\left\|\frac{m}{q}\right\| - \frac{1}{2q}} \\ &\leq 2\sum_{1 \leq m \leq \frac{q}{2}} \frac{1}{\frac{m}{q} - \frac{1}{2q}} \\ &= \sum_{1 \leq m \leq \frac{q}{2}} \frac{4q}{2m-1} \\ &\leq 4q \sum_{1 \leq m \leq q-1} \frac{1}{m} \\ &\leq 4q\log(2q).\end{aligned}$$

Otherwise, $1 \leq h \leq T/q$ or $\frac{q}{2} < r \leq q$, and then $hq + r \geq \frac{1}{2}(h+1)q$, and the sum over these indices is

$$\ll \sum_{0 \leq h \leq T/q} \sum_{1 \leq r \leq q} \min\left\{\frac{N}{(h+1)q}, \frac{1}{\left\|\frac{ra}{q} + hq\beta + r\beta\right\|}\right\}.$$

So we have got

$$\sum_{t \leq T} \min\left\{\frac{N}{t}, \frac{1}{\|t\alpha\|}\right\} \ll q\log(2q) + \sum_{0 \leq h \leq T/q} \sum_{1 \leq r \leq q} \min\left\{\frac{N}{(h+1)q}, \frac{1}{\left\|\frac{ra}{q} + hq\beta + r\beta\right\|}\right\}.$$

Let $1 \le h \le T/q$, let $I = [A, B]$ be a closed arc in $\mathbb{R}/\mathbb{Z}$ of measure q^{-1}, and let $J = [A - hq\beta - q^{-1}, B - hq\beta + q^{-1}] \subset \mathbb{R}/\mathbb{Z}$. For $1 \le r \le q$, if $\frac{ra}{q} + hq\beta + r\beta \in I$ then $\frac{ra}{q} + r\beta \in [A - hq\beta, B - hq\beta]$, and as $|r\beta| \le q^{-1}$, then $\frac{ra}{q} \in J$. As J is a closed arc with measure $3q^{-1}$ and $\frac{a}{q}, \frac{2a}{q}, \ldots, \frac{q \cdot a}{q}$ are distinct in $\mathbb{R}/\mathbb{Z}$, due to $\gcd(a, q) = 1$, there are at most four r, $1 \le r \le q$, for which $\frac{ra}{q} \in J$. Therefore there are at most four r, $1 \le r \le q$, for which $\frac{ra}{q} + hq\beta + r\beta \in I$.

For $0 \le j \le q - 1$, let $I_j = [jq^{-1}, (j+1)q^{-1}]$. If $\frac{ra}{q} + hq\beta + r\beta \in I_j \subset \mathbb{R}/\mathbb{Z}$, then

$$\left\| \frac{ra}{q} + hq\beta + r\beta \right\| \ge \min\{jq^{-1}, 1 - (j+1)q^{-1}\}$$

i.e.

$$\frac{1}{\left\| \frac{ra}{q} + hq\beta + r\beta \right\|} \le \frac{q}{\min\{j, q - j - 1\}}.$$

Therefore, for $1 \le r \le q$ with $\frac{ra}{q} + hq\beta + r\beta \in I_j \subset \mathbb{R}/\mathbb{Z}$,

$$\begin{aligned}
\min\left\{ \frac{N}{(h+1)q}, \frac{1}{\left\| \frac{ra}{q} + hq\beta + r\beta \right\|} \right\} &\le \min\left\{ \frac{N}{(h+1)q}, \frac{q}{\min\{j, q-j-1\}} \right\} \\
&\le \begin{cases} \frac{N}{(h+1)q} & j = 0, q-1 \\ \frac{q}{\min\{j, q-j-1\}} & 1 \le j \le q-2. \end{cases}
\end{aligned}$$

We have just established that for each $0 \le j \le q - 1$ there are at most four $1 \le r \le q$ such that $\frac{ra}{q} + hq\beta + r\beta \in I_j \subset \mathbb{R}/\mathbb{Z}$, and hence

$$\begin{aligned}
&\sum_{0 \le h \le T/q} \sum_{1 \le r \le q} \min\left\{ \frac{N}{(h+1)q}, \frac{1}{\left\| \frac{ra}{q} + hq\beta + r\beta \right\|} \right\} \\
&\le \sum_{0 \le h \le T/q} \sum_{0 \le j \le q-1} 4 \cdot \begin{cases} \frac{N}{(h+1)q} & j = 0, q-1 \\ \frac{q}{\min\{j, q-j-1\}} & 1 \le j \le q-2 \end{cases} \\
&= \sum_{0 \le h \le T/q} \left(\frac{8N}{(h+1)q} + \sum_{1 \le j \le q-2} \frac{q}{\min\{j, q-j-1\}} \right) \\
&\le \frac{8N}{q} \sum_{1 \le h \le \frac{T}{q} + 1} \frac{1}{h+1} + \left(\frac{T}{q} + 1 \right) \cdot 2q \sum_{1 \le j \le q/2} \frac{1}{j} \\
&\ll \frac{N}{q} \log(2T/q) + \left(\frac{T}{q} + 1 \right) \cdot q \log q.
\end{aligned}$$

Putting things together,

$$\sum_{t\le T}\min\left\{\frac{N}{t},\frac{1}{\|t\alpha\|}\right\}\ll q\log(2q)+\frac{N}{q}\log(2T)+\left(\frac{T}{q}+1\right)q\log(2q)$$
$$\ll q\log(2qT)+\frac{N}{q}\log(2qT)+T\log(2qT).$$

□

We now combine Theorem 5 and Theorem 6. For $U,V\ge 2, UV\le N, T\ge 1$, $\left|\alpha-\frac{a}{q}\right|\le\frac{1}{q^2}$, $\gcd(a,q)=1$,

$$\begin{aligned}|S_N(\alpha)|&\ll U+(\log N)\left(\frac{N}{q}+UV+q\right)\log(2qUV)\\&+N^{1/2}(\log N)^3\max_{U\le M\le N/V}\left(M+\left(\frac{N}{q}+\frac{N}{M}+q\right)\log(2qN/M)\right)^{1/2}\\&\ll U+(\log 2qN)^3\left(\frac{N}{q}+UV+q\right)\\&+N^{1/2}(\log qN)^{7/2}\max_{U\le M\le N/V}\left(M+\frac{N}{q}+\frac{N}{M}+q\right)^{1/2}\\&\ll U+(\log 2qN)^3\left(\frac{N}{q}+UV+q\right)\\&+N^{1/2}(\log qN)^{7/2}\left(\left(U+\frac{N}{q}+\frac{N}{U}+q\right)^{1/2}+\left(\frac{N}{V}+\frac{N}{q}+V+q\right)^{1/2}\right).\end{aligned}$$

Now take $U=V$, for which

$$\begin{aligned}|S_N(\alpha)|&\ll U+(\log 2qN)^3\left(\frac{N}{q}+U^2+q\right)\\&+N^{1/2}(\log qN)^{7/2}\left(U+\frac{N}{q}+\frac{N}{U}+q\right)^{1/2}\\&\ll U+(\log 2qN)^3\left(\frac{N}{q}+U^2+q\right)\\&+N^{1/2}(\log qN)^{7/2}(U^{1/2}+N^{1/2}q^{-1/2}+N^{1/2}U^{-1/2}+q^{1/2})\\&=U+(\log 2qN)^3\left(\frac{N}{q}+U^2+q\right)\\&+(\log qN)^{7/2}(N^{1/2}U^{1/2}+Nq^{-1/2}+NU^{-1/2}+N^{1/2}q^{1/2}).\end{aligned}$$

For $U=N^{2/5}$ we get the following.

Theorem 7. *There is some C such that if $\alpha\in\mathbb{R}$, $\left|\alpha-\frac{a}{q}\right|\le\frac{1}{q^2}$, $a\ge 1$, $\gcd(a,q)=1$, then for any $N\ge 1$,*

$$|S_N(\alpha)|\le C(Nq^{-1/2}+N^{4/5}+N^{1/2}q^{1/2})(\log N)^4.$$